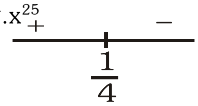
EXERCISE - 01 CHECK YOUR GRASP

1. $f(x) = x^{25}(1-x)^{75}$

$f'(x) = 25.x^{24}(1-x)^{75} - 75.(1-x)^{74}.x^{25}$

$= 25.x^{24}(1-x)^{74}\{1-x-3x\}$

$= 25x^{24}(1-x)^{74}(1-4x)$

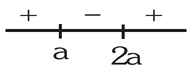


5. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1, \quad a > 0$

$f'(x) = 6x^2 - 18ax + 12a^2$

$= 6(x^2 - 3ax + 2a^2)$

$= 6(x-a)(x-2a)$



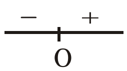
$p = a, \quad q = 2a$

$\Rightarrow \; a^2 = 2a$

$\Rightarrow \; a = 0$ (rejected) or $a = 2$

$a = 2$

6. $f'(x) = x(2^2 + 4^2.x^2 + 6^2.x^4 + ...+100^2.x^{98}\}$



7. $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$

$f'(x) = g(x) = 12x^3 - 12x^2 + 12x + a$

$f''(x) = 36x^2 - 24x + 12$

$= 12(3x^2 - 2x + 1)$

$f''(x) > 0$

$f''(x)$ is increasing

$\Rightarrow$ $f'(x) = 0$ at exactly one point.

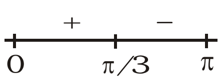
$\Rightarrow$ The given function has exactly one extremum.

10. $f(x) = \frac{1}{2}\{1 + \cos x\}\sin x$

$= \frac{\sin x}{2} + \frac{\sin 2x}{4}$

$f'(x) = \frac{\cos x}{2} + \frac{\cos 2x}{2}$

$= \frac{2\cos^2 x + \cos x - 1}{2} = \frac{(2\cos x - 1)(\cos x + 1)}{2}$



$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{8}$ = maximum value

$f(0) = f(\pi)$ = minimum value

11. Let the line be $(x-1) + \lambda(y-4) = 0$

$x + \lambda y = 1 + 4\lambda$

Sum of intercept $= 1 + 4\lambda + \frac{1+4\lambda}{\lambda}, \; \lambda > 0$

$= 4\lambda + \frac{1}{\lambda} + 5$

$\frac{4\lambda + \frac{1}{\lambda}}{2} \geq \sqrt{4\lambda.\frac{1}{\lambda}} \;\Rightarrow\; 4\lambda + \frac{1}{\lambda} \geq 4$

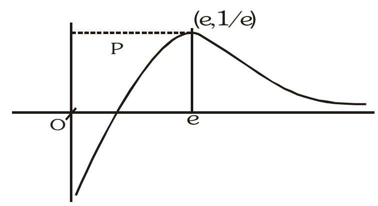
$\Rightarrow \quad 4\lambda + \frac{1}{\lambda} + 5 \geq 9$

equality holds for $4\lambda = \frac{1}{\lambda} \;\Rightarrow\; \lambda = \frac{1}{2}$

Required line is $x + \frac{y}{2} = 3$

$2x + y = 6$

12. $A = \frac{\ell n\, x}{x}$



$A' = \frac{1 - \ell n\, x}{x^2} = 0$ at $x = e$

$A'' = \frac{-x - 2x(2 - \ell n x)}{x^2}$

$A'' < 0$ at $x = e \Rightarrow$ maxima

$A_{max|x=e} = \frac{1}{e}$

16. $y = ax^3 + bx^2$

$y' = 3ax^2 + 2bx$

$y'' = 6ax + 2b$

for point of inflection $y'' = 0$

$x = \frac{-b}{3a}$

$3a + b = 0 \quad .....(i) \quad (\text{as } x = 1)$

point satisfy the curve also, so

$3 = a + b \quad .....(ii)$

from (i) & (ii)

$a = -\frac{3}{2}, \; b = \frac{9}{2}$

20. $f(x) = 2x^3 - 3(2+\lambda)x^2 + 12\lambda x$

$f'(x) = 6x^2 - 6(2+\lambda)x + 12\lambda$

$D > 0$

$36(2+\lambda)^2 - 24.12\,.\,\lambda > 0$

$\Rightarrow \; (\lambda - 2)^2 > 0$

$\Rightarrow \; \lambda \neq 2$

so required set is option (A,C,D)

1. $f(x) = x^3 - 3px^2 + 3(p^2-1)x + 1$

$f'(x) = 3\{x^2 - 2px + (p-1)(p+1)\}$

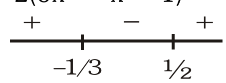
$f'(x) = 3\{x - (p-1)\}\{x - (p+1)\}$

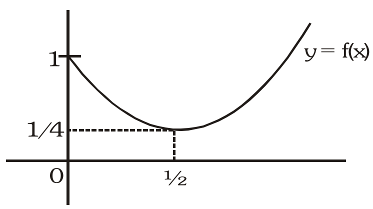
$-2 < p - 1 < 4$ and $-2 < p + 1 < 4$

$p \in (-1, 3)$

2. $f'(x) = 12x^2 - 2x - 2$

$= 2(6x^2 - x - 1) = 2(2x-1)(3x+1)$





$$\text{Min }\{f(t) : 0 \le t \le x\}\ ;\ 0 \le x \le 1 = \begin{cases} f(x), & 0 \le x \le \frac{1}{2} \\ f\left(\frac{1}{2}\right), & \frac{1}{2} < x \le 1 \end{cases}$$

4. The solution set of the inequality

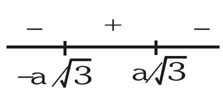
$$\frac{x^2+x+2}{x^2+5x+6} < 0 \Rightarrow -3 < x < -2$$

$f(x) = 1 + a^2x - x^3$

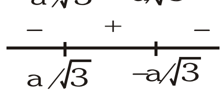
$f'(x) = a^2 - 3x^2$

$= (a - \sqrt{3}x)(a + \sqrt{3}x)$

If $a > 0$ $\quad -3 < \frac{-a}{\sqrt{3}} < -2$



If $a < 0$ $\quad -3 < \frac{a}{\sqrt{3}} < -2$



5. $f(x) = \int_0^x \sqrt{1-t^4}\,dt$

$f(-x) = \int_0^{-x} \sqrt{1-t^4}\,dt$

$= -\int_0^x \sqrt{1-u^4}\,du$ (Put $t = -u$)

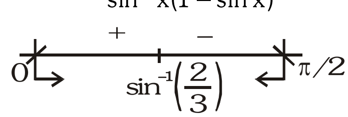
$f(-x) = -f(x) \Rightarrow$ 'f' is odd function.

Check other options.

8. $f(x) = \frac{4}{\sin x} + \frac{1}{1-\sin x}$ $\quad x \in \left(0, \frac{\pi}{2}\right)$

$f'(x) = \cos x\left\{\frac{1}{(1-\sin x)^2} - \frac{4}{\sin^2 x}\right\}$

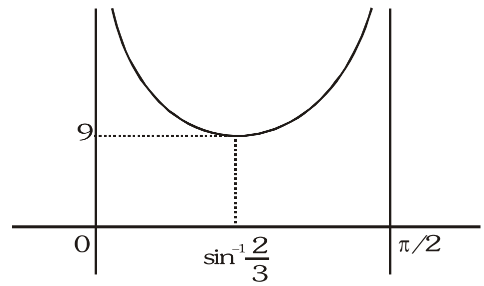
$= \cos x\,\frac{(2-\sin x)(3\sin x - 2)}{\sin^2 x(1-\sin x)^2}$



$f\left(\sin^{-1}\frac{2}{3}\right) = 6 + 3 = 9$

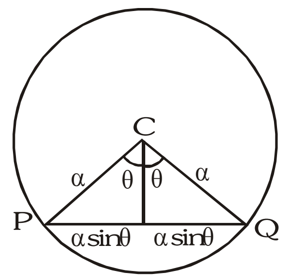
$f(0^+) \to \infty$

$f\left(\frac{\pi^-}{2}\right) \to \infty$



$\Rightarrow \quad a = 9$

11.



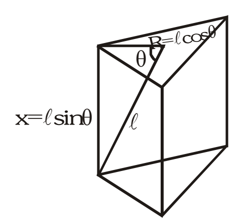
$s = \frac{2\alpha + 2\alpha\sin\theta}{2} = \alpha + \alpha\sin\theta$

$\Delta = \frac{1}{2}\alpha^2 \sin 2\theta$

$r = \frac{\Delta}{s} = \frac{1}{2}\alpha\left\{\frac{\sin 2\theta}{1+\sin\theta}\right\}$

Maximize 'r'.

12. Let 'a' be the side of base.



$\therefore \quad R = \frac{a}{\sqrt{3}}$

$\Rightarrow \quad a = \sqrt{3}.\ell\cos\theta$

volume = base area height

$= \left(\frac{\sqrt{3}}{4}.3\ell^2\cos^2\theta\right) \quad \ell\sin\theta$

$f(\theta) = \frac{3\sqrt{3}}{4}\ell^3\{\cos^2\theta\sin\theta\}$

$f'(\theta) = \frac{3\sqrt{3}\ell^3}{4}\{2\cos\theta.(-\sin^2\theta) + \cos^3\theta\}$

$= \frac{3\sqrt{3}\ell^3}{4}\{1 - 2\tan^2\theta\}\cos^3\theta$

for max. volume, $\tan\theta = \frac{1}{\sqrt{2}}$

altitude $= x = \ell.\sin\theta = \frac{\ell}{\sqrt{3}}$

Match the Column :

1. $y = ax^2 + bx + c$

$\because$ Points A, B and D lies on the curve.

$\therefore$ $4a - 2b + c = 3$

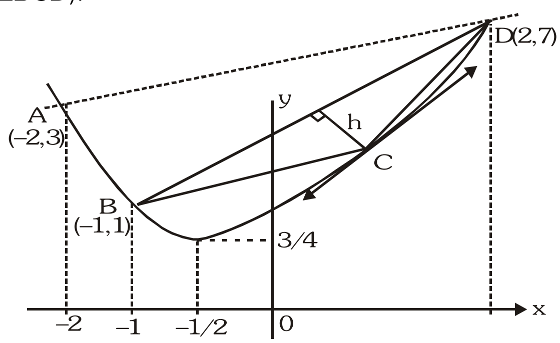
$a - b + c = 1$

$4a + 2b + c = 7$

Solving the equations we get $a = b = c = 1$.

$\therefore$ $y = x^2 + x + 1$

To maximize area of $\square$ABCD, we maximize area (ΔBCD).



To maximize Area(ΔBCD) we have to maximize h (as shown in figure)

for maximum h

$\Rightarrow$ Slope of BD = Slope of tangent at C

$$\frac{7-1}{2+1} = (2x + 1)$$

$$x = \frac{1}{2}$$

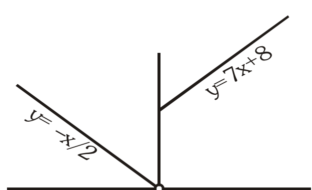
$$y = \frac{1}{4} + \frac{1}{2} + 1 = \frac{7}{4}$$

$$\therefore \quad C \equiv \left(\frac{1}{2}, \frac{7}{4}\right)$$

On the basis of this the coloumns can be matched.

Assertion and Reason :

1.

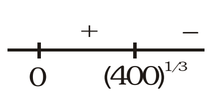


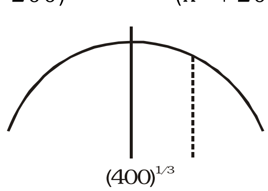
From figure st. I is false, because $f(0-h) < f(0)$

st. II is obviously true.

2. St. II :- $f(x) = \dfrac{x^2}{x^3+200}$

$$f'(x) = \frac{2x(x^3+200)-3x^4}{(x^3+200)^2} = \frac{x(400-x^3)}{(x^3+200)^2}$$





St. II is false.

St. I $\because$ f(x) has maxima at $x = (400)^{1/3}$ & 7 is the closest natural number.

$\therefore$ a_n has greatest value for n = 7.

Comprehension # 1 :

$$\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\ \ell n \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$$

$$\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\ \ell n \left(\frac{f(x)}{x^3}+1\right) = 2 \quad ....(1)$$

for limit to exist

$$\underset{x\to 0}{\text{Lt}}\ \frac{f(x)}{x^3} = 0$$

$\Rightarrow$ $f(x) = a_0x^6 + a_1x^5 + a_2x^4$

Also $f'(0) = f'(2) = f'(1) = 0$

$f'(x) = 6a_0x^5 + 5a_1x^4 + 4a_2x^3$

$= x^3(6a_0x^2 + 5a_1x + 4a_2)$

$f'(2) = 0$

$\Rightarrow$ $24a_0 + 10a_1 + 4a_2 = 0$ ......(2)

$f'(1) = 0$

$6a_0 + 5a_1 + 4a_2 = 0$ ....(3)

Consider eq^n. (1)

$$\ell n\left\{\underset{x\to 0}{\text{Lt}}\left(\frac{f(x)}{x^3}+1\right)^{\frac{1}{x}}\right\} = 2$$

$$\ell n\ e^{\left(\lim\limits_{x\to 0}\frac{f(x)}{x^4}\right)} = 2 \Rightarrow \underset{x\to 0}{\text{Lt}}\ \frac{a_0x^6+a_1x^5+a_2x^4}{x^4} = 2$$

$\Rightarrow$ $a_2 = 2$

Putting a_2 in (2) & (3)

$24a_0 + 10a_1 = -8$

$6a_0 + 5a_1 = -8$

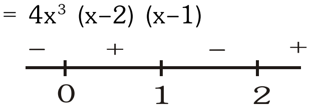
on solving this we get

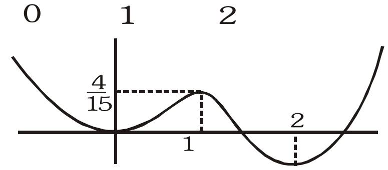
$$a_1 = -\frac{12}{5},\ a_0 = \frac{2}{3}$$

$$f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$$

$f'(x) = 4x^5 - 12x^4 + 8x^3 = 4x^3(x^2 - 3x + 2)$

$= 4x^3(x-2)(x-1)$





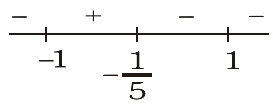
On the above basis the answers can be given.

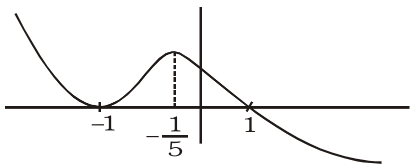
1. (b) $f(x) = -(x-1)^3 (x+1)^2$

$f'(x) = -\{3(x-1)^2(x+1)^2 + (x-1)^3\, 2(x+1)\}$

$= -(x-1)^2 (x+1) \{3x+3+2x-2\}$

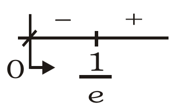
$= -(x-1)^2 (x+1)(5x+1)$





(c) $f(x) = x\,\ell n\, x$

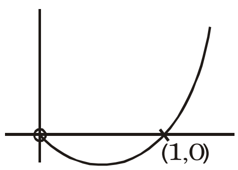
$f'(x) = 1 + \ell n\, x$



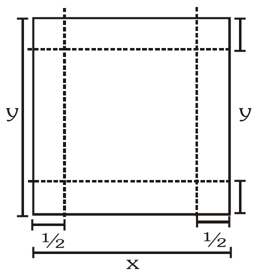
$f''(x) = \frac{1}{x} > 0$

$\Rightarrow$ concave up

$\underset{x\to 0^+}{\text{Lt}}\; x\,\ell n\, x = 0$, $\underset{x\to\infty}{\text{Lt}}\; x\,\ell n\, x \to \infty$



6.



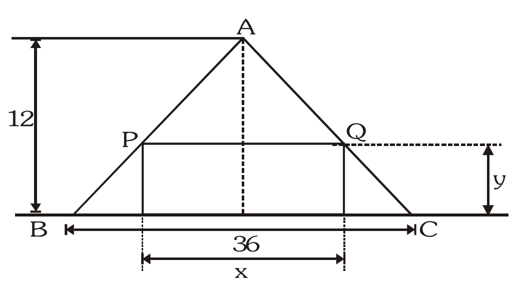
Given $xy = 18$

Printed area

$= f(x) = (x-1)\left(y - \frac{3}{2}\right) = (x-1)\left(\frac{18}{x} - \frac{3}{2}\right)$

Now maximize the area.

9.



$\Delta APQ \sim \Delta ABC$

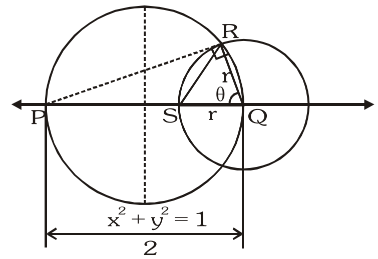
$\frac{12-y}{12} = \frac{x}{36}$

$3(12-y) = x$

$A = xy = 3(12-y)\, y$

Now maximize the area.

15.



$\cos\theta = \frac{r}{2}$

$r = 2\cos\theta$

$A(\Delta SRQ) = f(\theta) = \frac{1}{2}(2\cos\theta)^2.\sin\theta = 2\cos^2\theta\,\sin\theta$

Now maximize $f(\theta)$

12. $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$

$(R_3 \to R_3 - (R_1 + 2R_2))$

$f'(x) = 2ax + b \;\Rightarrow\; f(x) = ax^2 + bx + c$

$f(x)$ is maximum at $x = \frac{5}{2}$

$f'\left(\frac{5}{2}\right) = 0 \;\Rightarrow 5a + b = 0$

$f(0) = 2 \;\Rightarrow\; c = 2,\; f(1) = 1 \Rightarrow a + b + c = 1$

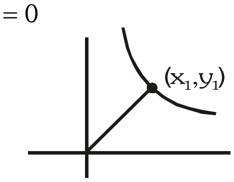
$\therefore\; a = \frac{1}{4},\; b = -\frac{5}{4},\; c = 2$

$f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

20. $ax^2 + 2bxy + ay^2 - c = 0$ ....(i)

$2xa + 2b\left(y + x\frac{dy}{dx}\right) + 2ay\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(2ax+2by)}{2bx+2ay}$



slope of normal $= \frac{bx+ay}{ax+by}$

slope of line joining origin & point $(x_1, y_1) = \frac{y_1}{x_1}$

minimum distance is along normal.

so $\frac{bx_1+ay_1}{ax_1+by_1} = \frac{y_1}{x_1} \;\Rightarrow\; x_1^2 = y_1^2$

$\Rightarrow\; x_1 = y_1$ or $x_1 = -y_1$ ....(ii)

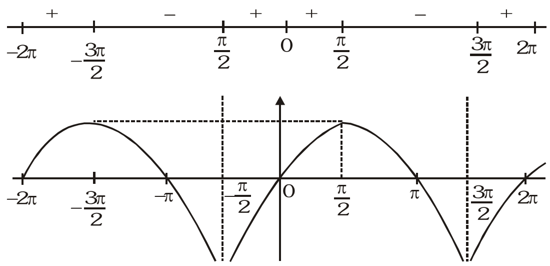
from (i) & (ii) required points are

for $x_1 = y_1$; $\left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right) \& \left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

for $x_1 = -y_1$ $\left(\pm\sqrt{\frac{c}{2(a-b)}}, \mp\sqrt{\frac{c}{2(a-b)}}\right)$ not possible

since $a-b<0$

2. $f'(x) = \dfrac{\cos x}{1+\sin x}$



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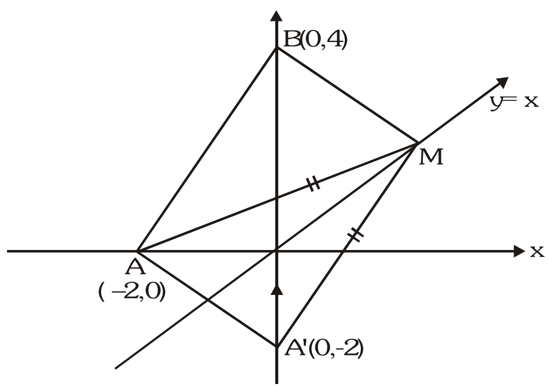


To minimize the perimeter.

AM + MB is to be minimized.

i.e. A'M + MB is to be minimized.

[where A' is image of A in y = x.]

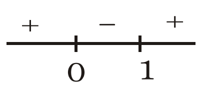
Obviously A'M + MB is minimized.

when A',M and B are collinear.

i.e. M coincide with origin.

$\therefore$ M $\equiv$ (0, 0)

4. $f(x) = x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2} - \log_{1/4}(m)$



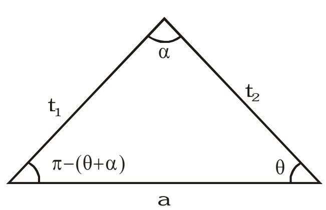
$f'(x) = 3x^2 - 3x = 3x(x-1)$

for f(x) = 0 to have 3 real & distinct roots

$f(0).f(1) < 0$

Solving this we get the required set of m.

9.



$$\frac{a}{\sin\alpha} = \frac{t_1}{\sin\theta} = \frac{t_2}{\sin(\theta+\alpha)}$$

$t_1 = \dfrac{a}{\sin\alpha}\sin\theta$, $t_2 = \dfrac{a}{\sin\alpha}\sin(\theta+\alpha)$

to maximize perimeter we maximize $t_1 + t_2$

$$t_1 + t_2 = f(\theta) = 2\frac{a}{\sin\alpha}\{\sin(\theta) + \sin(\theta+\alpha)\}$$

$$= \frac{a}{\sin\alpha}\left\{\sin\left(\theta+\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}\right\}$$

for maximum $\sin\left(\theta+\dfrac{\alpha}{2}\right) = 1$

12. $f(x) = \sin^3x + \lambda\sin^2x$

$f'(x) = \sin x\cos x(3\sin x + 2\lambda)$

$f''(x) = 6\sin x\cos^2x - 3\sin^3x + 2\lambda\cos 2x$

$f'(x) = 0 \Rightarrow \sin x = 0$ or $\cos x = 0$ or $\sin x = \dfrac{-2\lambda}{3}$

$\cos x \neq 0$ if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\sin x = 0 \Rightarrow x = 0$

$\sin x = \dfrac{-2\lambda}{3}$

$-1 < \sin x < 1 \Rightarrow -1 < \dfrac{-2\lambda}{3} < 1$

$\Rightarrow \dfrac{-3}{2} < \lambda < \dfrac{3}{2}$

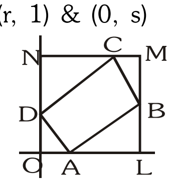
$\lambda \neq 0$ otherwise there is only one critical point.

If $\lambda > 0$, then $f''(0) > 0 \Rightarrow x = 0$ point of minima & f'(x) changes sign from positive to negative for

$x = \sin^{-1}\left(\dfrac{-2\lambda}{3}\right)$ (point of maxima).

If $\lambda < 0$ then x = 0 is a point of maxima while $x = \sin^{-1}\left(\dfrac{-2\lambda}{3}\right)$ is a point of minima. Thus for $\lambda \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right) - \{0\}$ function has exactly one maxima & exactly one minima.

16. Let the vertices L, M, N of the square S be (1, 0), (1, 1) & (0, 1) respectively & the vertex O be origin. Let the co-ordinate of vertices A, B, C, D of the quadrilateral be (p, 0)(1, q)(r, 1) & (0, s)



Then $a^2 = (1-p)^2 + q^2$

$b^2 = (1-q)^2 + (1-r)^2$

$c^2 = (1-s)^2 + r^2$

$d^2 = p^2 + s^2$

Thus $a^2 + b^2 + c^2 + d^2 = (1-p)^2 + q^2 + (1-q)^2 + (1-r)^2 + (1-s)^2 + r^2 + p^2 + s^2$

Let $f(x) = x^2 + (1-x)^2 \quad 0 \le x \le 1$

$f'(x) = 2x - 2(1-x)$

$f'(x) = 0 \Rightarrow x = 1/2$

$f''(x) = 4$

$\Rightarrow$ f(x) is minimum at x = 1/2 & max. value of f(x) occur at x = 0, x = 1

$\therefore 1/2 \le f(x) \le 1$

So $2 \le a^2 + b^2 + c^2 + d^2 \le 4$

1. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1 \quad a > 0$

$\therefore$ $f'(x) = 6x^2 - 18ax + 12a^2$

$\therefore$ $f''(x) = 12x - 18a$

for maximum or minimum

$6x^2 - 18ax + 12a^2 = 0$

$x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$

maximum at $x = a$ and minimum at $x = 2a$

$\therefore$ ($a > 0$) given)

$p = a, \quad q = 2a$

$\therefore$ $p^2 = q$

$a^2 = 2a$

$a(a - 2) = 0$

$a = 2$

2. $f(x) = x + \frac{1}{x} \qquad f'(x) = 1 - \frac{1}{x^2}$

$x = \pm 1$

$f''(x) = \frac{2}{x^3}$

minimum at $x = 1$

3. $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$u^2 = a^2 + b^2 + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta)}$

$u^2 = a^2 + b^2 + 2\sqrt{\begin{array}{l}a^4\cos^2\theta\sin^2\theta + a^2b^2\cos^4\theta \\ +a^2b^2\sin^4\theta + b^4\sin^2\theta\cos^2\theta\end{array}}$

$u^2 = a^2 + b^2 + 2\sqrt{\begin{array}{l}a^2b^2(1 - 2\sin^2\theta\cos^2\theta) \\ +a^4\cos^2\theta\sin^2\theta + b^4\cos^2\theta\sin^2\theta\end{array}}$

$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^4 - b^4 - 2a^2b^2)\sin^2\theta\cos^2\theta}$

$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^2 - b^2)^2 \times \left(\frac{\sin 2\theta}{2}\right)^2}$

$= a^2 + b^2 + \sqrt{4a^2b^2 + (a^2 - b^2)^2\sin^2 2\theta}$

u^2 is maximum when $\sin^2 2\theta = 1$

u^2 is minimum when $\sin^2 2\theta = 0$

$u^2_{(max.)} - u^2_{(min.)}$

$2(a^2 + b^2) - (a + b)^2$

$2a^2 + 2b^2 - a^2 - b^2 - 2ab$

$a^2 + b^2 - 2ab = (a - b)^2$

4. $f(x) = \frac{x}{2} + \frac{2}{x}$

$\Rightarrow \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$

For maximum or minimum, $f'(x) = 0$

$\frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow \quad x^2 = 4 \qquad \Rightarrow \quad x = \pm 2$

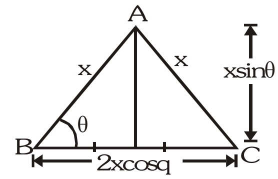
Now, $f''(x) = \frac{4}{x^3}$

at $\quad x = 2, \qquad f''(x) > 0$

and $\quad x = -2 \qquad f''(x) < 0$

So, there exists a local minimum at $x = 2$.

5. A triangular park



$\Delta = \frac{1}{2}(2x\cos\theta)(x\sin\theta)$

$= \frac{1}{2}x^2\sin 2\theta$

$\boxed{\Delta_{max.} = \frac{x^2}{2}}$

6. Using A.M. $\geq$ G.M.

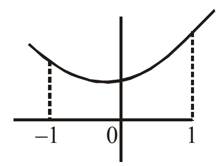
$\frac{p^2 + q^2}{2} \geq p.q$

$\Rightarrow \quad pq \leq \frac{1}{2}$

$\Rightarrow \quad (p + q)^2 = p^2 + q^2 + 2pq$

$\Rightarrow \quad (p + q) \leq \sqrt{2}$

8. Graph of P(x) under given conditions. It is clear that P(x) has max. at 1 but not minimum at –1.



9. Point (t^2, t) is on the parabola $x = y^2$

Its distance from $y - x = 1$

$d(t) = \frac{t^2 - t + 1}{\sqrt{2}}$

$d'(t) = \frac{1}{\sqrt{2}}[2t - 1] = 0$

$t = \frac{1}{2}$

$d''(t) = \frac{2}{\sqrt{2}} > 0$

$d(t)$ is min at $t = \frac{1}{2}$

Its value

$d\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{1}{2} + 1\right)$

$d\left(\frac{1}{2}\right) = \frac{3\sqrt{2}}{8}$

10. f has a local minimum at $x = -1$

$\therefore \lim_{x \to -1} f(x) \geq f(-1)$

$k + 2 \leq 1$

$k \leq -1$

$\therefore$ $k = -1$

Directions : Questions number 86 to 90 are Assertion - Reason type questions. Each of these questions contains two statements :

Statement–1 (Assertion) and Statement–2 (Reason).
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice.

11. $f'(x) = \sqrt{x}\sin x$

$f'(\pi)$ & $f'(2\pi)$ are 0.

$$f'(x) \quad \underset{\pi}{\overset{+}{}} \;\; - \;\; \underset{2\pi}{} \;\; +$$

$\Rightarrow$ local maximum at $x = \pi$ and local minimum at $x = 2\pi$

13. At $x = 0$ $\quad f(x) = 1$
and for $x = h$ and $x = -h$ $\quad (h \to 0;\ h > 0)$

$$\frac{\tan x}{x} > 1$$

$\therefore$ Function has a minima at $x = 0$
$\therefore$ Statement–1 is true.

Now $f(x) = \begin{cases} \dfrac{\tan x}{x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$

$$f'(x) = \begin{cases} \dfrac{x\sec^2 x - \tan x}{x^2} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f'(0) = 0$
$\therefore$ Statement-2 is also true.

14. $V = \dfrac{4}{3}\pi r^3$

Initially $r = 4500\ \pi$, $r = r_0$

$$4500\ \pi = \frac{4}{3}\pi\ r_0^3 \Rightarrow \boxed{r_0 = 15\text{m}}$$

Now $\dfrac{dV}{dt} = \dfrac{4}{3}\pi\ (3r^2)\dfrac{dr}{dt}$

$$-72\ \pi = 4\pi\ r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-18}{r^2} \quad \text{......(i)}$$

$$\int r^2 dr = -\int 18\ dt \Rightarrow \frac{r^3}{3} = -18\ t + C$$

At $t = 0$, $r = 15$ m

So, $\dfrac{(15)^3}{3} = -18(0) + C \Rightarrow C = 1125$

$\Rightarrow r^3 = -54t + 3375$ ......(ii)

At time $t = 49$ min $\quad r = 9$ m
from eq. (i)

$$\left(\frac{dr}{dt}\right)_{t=49} = \frac{-18}{(9)^2} = -2/9$$

(Negative sign shows decrement in radii)

15. $f'(x) = \dfrac{1}{x} + 2bx + a$

$f'(-1) = -1 - 2b + a = 0$ ......(1)

$f'(2) = \dfrac{1}{2} + 4b + a = 0$ ...... (2)

solve (1) & (2) $\Rightarrow a = \dfrac{1}{2}$, $\quad b = -\dfrac{1}{4}$

$\therefore$ st : 2 is true

$f''(x) = -\dfrac{1}{x^2} - \dfrac{1}{2} = -\left(\dfrac{1}{x^2} + \dfrac{1}{2}\right)$ $\quad$ (always –ive)

$f''(-1) = -\dfrac{3}{2} < 0$

$f''(2) = -\dfrac{3}{4} < 0$

$\therefore$ Local maximum at $x = -1$ & 2

16. $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0$

$\Rightarrow$ f is increasing function

$\Rightarrow$ $f(x) = 0$ has exactly one real root

(as it is an odd degree polynomial)

2. $f(x) = (1 + b^2)x^2 + 2bx + 1$

It is a quadratic expression with coeff. of $x^2 = 1 + b^2 > 0$.

$\therefore$ $f(x)$ represents an upward parabola whose min value is $\frac{-D}{4a}$, D being the discriminant.

$\therefore$ $m(b) = -\frac{4b^2 - 4(1+b^2)}{4(1+b^2)} \Rightarrow m(b) = \frac{1}{1+b^2}$

For range of m(b) :

$\frac{1}{1+b^2} > 0$ also $b^2 \ge 0 \Rightarrow 1 + b^2 \ge 1$

$\Rightarrow \frac{1}{1+b^2} \le 1$

Thus m(b) = (0, 1]

7. Let $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = 10$

$\Rightarrow -a + b - c + d = 10$ ......(i)

$p(1) = -6$

$\Rightarrow a + b + c + d = -6$......(ii)

p(x) has maxima at x = – 1

$\therefore p'(-1) = 0$

$\Rightarrow 3a - 2b + c = 0$ .....(iii)

p'(x) has min. at x = 1

$\therefore p''(1) = 0$

$\Rightarrow 6a + 2b = 0$ .....(iv)

Solving (i), (ii), (iii) and (iv) we get

From (iv) $b = -3a$

From (iii) $3a + 6a + c = 0 \Rightarrow c = -9a$

From (ii) $a - 3a - 9a + d = -6 \Rightarrow d=11a - 6$

From (i) $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow 16a = 16 \Rightarrow a = 1$

$\Rightarrow b = -3, c = -9, d = 5$

$\therefore p(x) = x^3 - 3x^2 - 9x + 5$

$\Rightarrow p'(x) = 3x^2 - 6x - 9 = 0$

$\Rightarrow 3(x + 1)(x - 3) = 0$

$\Rightarrow$ x = –1 is a pt. of max (given) and x = 3 is at pt. of min.

[$\because$ max and min occur alternatively]

$\therefore$ pt. of local max is (–1, 10) and pt. of local min is (3, –22)

And distance between them is

$= \sqrt{[3-(-1)]^2 + (-22-10)^2} = \sqrt{16+1024}$

$= \sqrt{1040} = 4\sqrt{65}$

9. (a,b) $\because$ $g(x) = \int_0^x f(t)\,dt$

$\Rightarrow g'(x) = f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$

$\therefore g'(x) = 0$

at $x = 1 + \ell n2$

$x = 0$ & $x = e$

$g''(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{cases}$

$\therefore$ $g''(1 + \ln 2) = -2$ and $g''(e) = 1 \Rightarrow g(x)$ has local max. at $x = 1 + \ln 2$ and local min. at $x = e$.

11. (A) $y = \frac{x^2 + 2x + 4}{x+2}$

$\Rightarrow x^2 + (2 - y)x + 4 - 2y = 0$

x is real ; so $D \ge 0$

$y^2 + 4y - 12 \ge 0$

$y \le -6, y \ge 2$

so minimum value = 2

(B) $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow AB = BA$

as A is symmetric & B is skew symmetric

$\Rightarrow (AB)^t = -AB$

$\Rightarrow k = 1, 3$

(C) $a = \log_3\log_3 2 \Rightarrow 3^{-a} = \log_2 3$

Now $1 < 2^{(-k+3^{-a})} < 2$

$\Rightarrow 1 < 2^{(-k+\log_2 3)} < 2 \Rightarrow 1 < 3.2^{-k} < 2$

$\Rightarrow \frac{1}{3} < 2^{-k} < \frac{2}{3} \Rightarrow \frac{3}{2} < 2^k < 3$

so k = 1 is possible

(D) $\sin\theta = \cos\phi$

$\Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = \cos\phi$

$\frac{\pi}{2} - \theta = 2n\pi \pm \phi$

$\frac{\pi}{2} - \theta = 2n\pi \pm \phi$

$\Rightarrow \theta \pm \phi - \frac{\pi}{2} = -2n\pi$

$\Rightarrow \frac{1}{\pi}(\theta \pm \phi - \frac{\pi}{2})$ = even integer

12. $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$

$f'(x) = \dfrac{2a(x^2-1)}{(x^2+ax+1)^2}$ and $f''(x) = \dfrac{4a(-x^3+3x+a)}{(x^2+ax+1)^3}$

$f''(1) = \dfrac{4a}{(a+2)^2}$ and $f''(-1) = \dfrac{-4a}{(a-2)^2}$

$\therefore\ (a+2)^2 f''(1) + (2-a)^2 f''(-1) = 0$

13. As when $x \in (-1, 1)$, $f'(x) < 0$

so $f(x)$ is decreasing on $(-1, 1)$ at $x = 1$

$f''(1) = \dfrac{4a}{(a+2)^2} > 0$ so local minima at $x = 1$.

14. $g(x) = \displaystyle\int_0^{e^x} \frac{f'(t)}{1+t^2}\,dt$

$g'(x) = \dfrac{f'(e^x)}{1+e^{2x}} e^x = \dfrac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})}$

$g'(x) > 0$ when $x > 0$

$g'(x) < 0$ when $x < 0$

15. $f(x) = 2x^3 - 15x^2 + 36x - 48$

Set $A = \{x \mid x^2 + 20 \le 9x\}$

$\because\ x^2 - 9x + 20 \le 0$

$(x-5)(x-4) \le 0$

$\Rightarrow\ x \in [4, 5]$

Now, $f'(x) = 6x^2 - 30x + 36 = 0$

$\Rightarrow\ x^2 - 5x + 6 = 0$

$x = 2, 3$ and $f(x) \uparrow$ in $x \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ In the set A, $f(x)$ is increasing

$\Rightarrow\ f(x)_{max} = f(5)$

$= 2.125 - 15.25 + 36.5 - 48$

$= 7$

16. $\displaystyle \operatorname*{Lt}_{x\to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2$

$\Rightarrow\ \displaystyle \operatorname*{Lt}_{x\to 0} \frac{p(x)}{x^2} = 1$

Let $p(x) = ax^4 + bx^3 + cx^2$

$\because\ \displaystyle \operatorname*{Lt}_{x\to 0} \frac{p(x)}{x^2} = 1 \Rightarrow c = 1$

$p(x) = ax^4 + bx^3 + x^2$

Now, $p'(x) = 4ax^3 + 3bx^2 + 2x$

$\because\ p'(1) = 0,\ p'(2) = 0$

$\Rightarrow\ 4a + 3b + 2 = 0$

$32a + 12b + 4 = 0 \Rightarrow a = \dfrac{1}{4},\ b = -1$

$\Rightarrow\ p(x) = \dfrac{1}{4}x^4 - x^3 + x^2 \Rightarrow p(2) = 4 - 8 + 4 = 0$

17. If $x \in [0, 1]$

then $x^2 \le x \le 1$

$x^2 e^{x^2} \le x e^{x^2} \le e^{x^2}$

Add e^{-x^2} to all sides

$x^2 e^{x^2} + e^{-x^2} \le x e^{x^2} + e^{-x^2} \le e^{x^2} + e^{-x^2}$

$\Rightarrow\ h(x) \le g(x) \le f(x)$ .... (i)

where, $f(x) = e^{x^2} + e^{-x^2}$

$f'(x) = 2x(e^{x^2} - e^{-x^2}) > 0$

$\Rightarrow$ $f(x)$ has a maxima at $x = 1$

$\Rightarrow\ a = e + \dfrac{1}{e}$

$h(x) = x^2 e^{x^2} + e^{-x^2}$

$h'(x) = 2x^3 e^{x^2} + 2x e^{x^2} - 2x e^{-x^2}$

$= 2x^3 e^{x^2} + 2x(e^{x^2} - e^{-x^2}) > 0$

$\Rightarrow$ $h(x)$ has a maxima at $x = 1$

$\Rightarrow\ c = e + \dfrac{1}{e}$

$\because\ h(x) \le g(x) \le f(x)$

$\Rightarrow$ $g(x)$ also has a maximum value at $x = 1$

$\Rightarrow\ a = b = c$

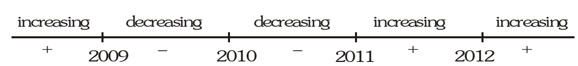
18. $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$

$\forall\ x \in R$

$f(x) = \ell n(g(x))\ \forall\ x \in R$

$g(x) = e^{f(x)}$

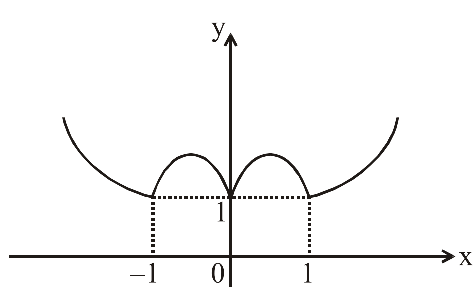
$g'(x) = 0 \Rightarrow e^{f(x)}.f'(x) = 0 \Rightarrow f'(x) = 0$



local maximum at $x = 2009$, hence only 1 point.

19. $f(x) = |x| + |(x+1)(x-1)|$

$$\Rightarrow\ f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x < 0 \\ -x^2 + x + 1 & 0 \le x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$



$\therefore$ f has 5 points where it attains either a local maximum or local minimum.

20. Let $P'(x) = k(x - 1)(x - 3)$

$= k(x^2 - 4x + 3)$

$\Rightarrow P(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$

$\because P(1) = 6$

$\Rightarrow \frac{4k}{3} + c = 6$ ....(1)

$P(3) = 2$

$\Rightarrow c = 2$ ....(2)

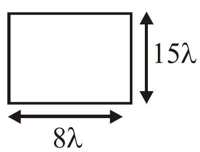
by (i) and (ii)

$k = 3$

$\therefore P'(x) = 3(x - 1)(x - 3)$

$\Rightarrow P'(0) = 9$

21. Where $P = 8\lambda + 15\lambda + 8\lambda + 15\lambda$ & λ is constant



Let removed length from each sides is x

Removed area is $4x^2 = 100 \Rightarrow x = 5$

$V = (8\lambda - 2x)(15\lambda - 2x)x$

$V = 120l^2x - 46\lambda x^2 + 4x^3$

$\frac{dv}{dx} = 120\lambda^2 - 92\lambda x + 12x^2 = 0$

Put $x = 5 \Rightarrow$ $120\lambda^2 - 460\lambda + 300 = 0$

$12\lambda^2 - 40\lambda + 30 = 0$

$6\lambda^2 - 23\lambda + 15 = 0$

$(\lambda - 3)(6\lambda - 5) = 0$

$\lambda = 3$ & $\lambda = \frac{5}{6}$

$\frac{d^2v}{dx^2} = -92\lambda + 24x = 120 - 92\lambda$

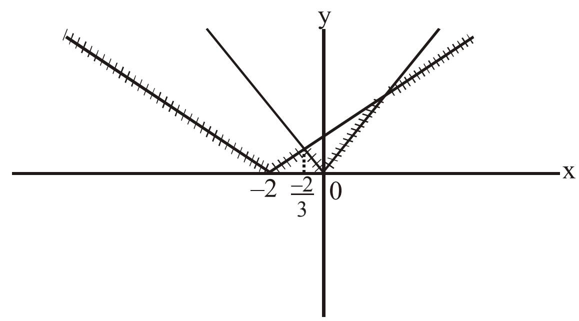
at $\lambda = 3 \Rightarrow \frac{d^2v}{dx^2} < 0$

at $\lambda = \frac{5}{6} \Rightarrow \frac{d^2v}{dx^2} > 0$ (rejected)

22. $f(x) = (a + b) - |b - a|$

$= \begin{cases} 2a, & a \le b \\ 2b, & a > b \end{cases} = 2\min(a, b)$

where $a = 2|x|$, $b = |x + 2|$



$\therefore$ Local maxima and minima at $x = -2, -\frac{2}{3}$ & 0