TRIGONOMETRIC EQUATION

EXERCISE - 01 CHECK YOUR GRASP

3. $\tan^2x - \sec^{10}x + 1 = 0$

$\Rightarrow \sec^2x - 1 - \sec^{10}x + 1 = 0$

$\Rightarrow \sec^2x\,(1 - \sec^8x) = 0$

$\Rightarrow \sec^8x = 1 \quad \Rightarrow \quad \cos x = \pm 1$

$\Rightarrow x = \pi, 2\pi, 3\pi$

4. $(\cos\theta + \cos 2\theta)^3 = \cos^3\theta + \cos^3 2\theta$

$\Rightarrow 3\cos\theta \cos 2\theta\,(\cos\theta + \cos 2\theta) = 0$

$\Rightarrow 6\cos\theta \cos 2\theta \cos\frac{3\theta}{2} \cos\frac{\theta}{2} = 0$

$\Rightarrow \theta = (2n+1)\frac{\pi}{2},\ \theta = (2m+1)\frac{\pi}{4},$

$\theta = (2p+1)\frac{\pi}{3},\ \theta = (2q+1)\pi$

$\therefore$ least positive value is $\frac{\pi}{4}$

8. $\tan(\alpha + \beta) = 3, 0 < \alpha + \beta < 90$

$\tan(\alpha - \beta) = 2,\ 0 < \alpha - \beta < 90$

$\therefore \sin(2\alpha) = \sin[(\alpha + \beta) + (\alpha - \beta)]$

$= \sin(\alpha + \beta)\cos(\alpha - \beta) + \cos(\alpha + \beta)\sin(\alpha - \beta)$

$= \frac{3}{\sqrt{10}}.\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{10}}.\frac{2}{\sqrt{5}} = \frac{1}{\sqrt{2}}$

11. $\sin x + 3\sin 2x + \sin 3x = \cos x + 3\cos 2x + \cos 3x$

$\Rightarrow 2\sin 2x\cos x + 3\sin 2x = 2\cos 2x\cos x + 3\cos 2x$

$\Rightarrow \sin 2x\,[2\cos x + 3] = \cos 2x\,[2\cos x + 3]$

$\Rightarrow \tan 2x = 1$

$\Rightarrow 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$

$\therefore x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{13\pi}{8}, \frac{9\pi}{8}$

12. $1 + \sin^4x = \cos^2 3x$

L.H.S. ≥ 1 and R.H.S. ≤ 1

So equality holds only if LHS = RHS

$\sin^4x = 0,\ \cos^2 3x = 1$

$x = n\pi, \quad \cos 3x = 1$ or $\cos 3x = -1$

$3x = 2n\pi \pm 0$ or $3x = 2n\pi \pm \pi$

$x = \frac{2n\pi}{3},\ x = \frac{2}{3}n\pi \pm \frac{\pi}{3}$

so greatest positive solution is 2π

13. $2^{\sin^2 x} + 4 . 2^{\cos^2 x} = 6$

Let $2^{\sin^2 x} = t$

$\therefore \quad t + \frac{8}{t} = 6 \Rightarrow (t-4)(t-2) = 0$

$2^{\sin^2 x} = 4$ & $2^{\sin^2 x} = 2$

$\Rightarrow \sin x = \pm\sqrt{2}$ (reject) and $\sin x = \pm 1$

$\therefore$ in $(-2\pi, 2\pi)$ there are 4 solutions.

14. $|\sin x| = \cos x, \quad \cos x > 0$

$\Rightarrow \sin^2x = \cos^2 x \quad \Rightarrow \quad \cos 2x = 0$

$\Rightarrow x = 2n\pi \pm \frac{\pi}{4}$

20. $|4\sin x - 1| < \sqrt{5} \quad \Rightarrow \quad -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$\Rightarrow \quad -\left(\frac{\sqrt{5}-1}{4}\right) < \sin x < \left(\frac{\sqrt{5}+1}{4}\right)$

$\Rightarrow \quad -\sin\frac{\pi}{10} < \sin x < \cos\frac{\pi}{5}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\frac{3\pi}{10}$

$\Rightarrow \quad x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$

EXERCISE - 02 BRAIN TEASERS

1. $\cos^2x + \cos^2 2x + \cos^2 3x = 1$

$\Rightarrow \cos^2x - \sin^2 2x + \cos^2 3x = 0$

$\Rightarrow \cos x \cos 3x + \cos^2 3x = 0$

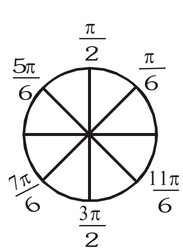
$\Rightarrow \cos 3x\,(\cos x + \cos 3x) = 0$

$\Rightarrow \cos 3x \cos 2x \cos x = 0$

$\Rightarrow x = (2p+1)\frac{\pi}{6}$

$x = (2q+1)\frac{\pi}{4},$

$x = (2r+1)\frac{\pi}{2},\ n \in I$



$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}$ also satisfy the equation.

3. $\log_{\left(x+\frac{1}{x}\right)}\left(2\sin\alpha - 1\right) \leq 0$

$\because \quad x + \frac{1}{x} > 2$

so $\quad 2\sin\alpha - 1 \leq 1 \quad \& \quad 2\sin\alpha - 1 > 0$

$\Rightarrow \quad 2\sin\alpha \leq 2 \quad \& \quad \sin\alpha > \frac{1}{2}$

$\Rightarrow \quad \sin\alpha \leq 1$

$\alpha \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

5. $\sin(\theta + \alpha) = k \sin 2\theta$

$\sin\theta \cos\alpha + \cos\theta \sin\alpha = 2k \sin\theta \cos\theta$

Change the equation in $\tan\frac{\theta}{2}$ form and let

$\tan\frac{\theta}{2} = t$, then obtained equation is

$\sin\alpha t^4 - (2\cos\alpha + 4k)t^3 + t(4k - 2\cos\alpha) - \sin\alpha = 0$

$S_1 = \frac{2\cos\alpha + 4k}{\sin\alpha}$, $S_2 = 0$

$S_3 = \frac{4k - 2\cos\alpha}{\sin\alpha}$, $S_4 = -1$

$\tan\frac{(\theta_1 + \theta_2 + \theta_3 + \theta_4)}{2} = \frac{S_1 - S_3}{1 - S_2 + S_4} = \infty$

$\Rightarrow \frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1)\pi$, $n \in$ integer

8. $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$

$\sin 2\theta - 2\cos\theta + 3 - 4\sin\theta - (1 - 2\sin^2\theta) = 0$

$2\cos\theta(\sin\theta - 1) + 2\sin^2\theta - 4\sin\theta + 2 = 0$

$\cos\theta(\sin\theta - 1) + (\sin\theta - 1)^2 = 0$

$(\sin\theta - 1)(\cos\theta + \sin\theta - 1) = 0$

$\sin\theta = 1$ or $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\theta = 2n\pi + \frac{\pi}{2}$ or $\theta - \frac{\pi}{4} = 2m\pi + \frac{\pi}{4}, 2m\pi - \frac{\pi}{4}$

$\theta = 2m\pi + \frac{\pi}{2}, 2m\pi$

12. $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$

$\Rightarrow$ $2\sin x - \sin 2x - 2\cos 2x - 2\sin x \sin 2x + 2\cos x = 0$

$\Rightarrow$ $2\sin x - \sin 2x - 2\cos 2x - (\cos x - \cos 3x) + 2\cos x = 0$

$\Rightarrow$ $2\sin x(1 - \cos x) + 4\cos^3 x - 3\cos x + \cos x - 2(2\cos^2 x - 1) = 0$

$\Rightarrow$ $2\sin x(1 - \cos x) + 4\cos^3 x - 4\cos^2 x - 2\cos x + 2 = 0$

$\Rightarrow$ $2\sin x(1 - \cos x) - 4\cos^2 x(1 - \cos x) + 2(1 - \cos x) = 0$

$\Rightarrow$ $(1 - \cos x)\{2\sin x - 4(1 - \sin^2 x) + 2\} = 0$

$\Rightarrow$ $\cos x = 1$ or $\sin x - 2(1 - \sin^2 x) + 1 = 0$

$\Rightarrow$ $x = 2n\pi$ $(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = 2n\pi$, $x = n\pi + (-1)^n\frac{\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}$

13. $2\cos^3 3\theta + 3\cos 3\theta + 4 = 3\sin^2 3\theta$

$\Rightarrow \left(\cos 3\theta + \frac{1}{2}\right)\left(2\cos^2 3\theta + 2\cos 3\theta + 2\right) = 0$

$\Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = 2n\pi \pm \frac{2\pi}{3}$

$\Rightarrow \theta = \frac{2n\pi}{3} \pm \frac{2\pi}{9}$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

True/False :

1. $\therefore \sin\left(\frac{\pi}{2} - \sin\theta\right) > \sin(\cos\theta)$

$\frac{\pi}{2} - \sin\theta > \cos\theta$, $\theta \in \left[0, \frac{\pi}{2}\right]$

Fill in the blanks :

2. $\cos x + \cos y = \frac{3}{2}$

$\Rightarrow \cos x + \cos\left(\frac{2\pi}{3} - x\right) = \frac{3}{2}$ $\left(x + y = \frac{2\pi}{3}\right)$

$\Rightarrow \cos x - \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{3}{2}$

$\Rightarrow \frac{\cos x + \sqrt{3}\sin x}{2} = 3$

$\Rightarrow$ No solution $\sin\left(x + \frac{\pi}{3}\right) = 3$

Match the column :

1. (A) $|\tan x| = \frac{m}{n} \Rightarrow \tan x = \frac{m}{n}$ & $\tan x = -\frac{m}{n}$

In $[0, 2\pi]$ it has 4 solutions

(B) $\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$

$\Rightarrow$ $\cos x = \cos 2x = \cos 3x = \cos 4x = \cos 5x = 1$

$\Rightarrow$ $x = 2n_1\pi$, $x = n_2\pi$,

$x = 2n_3\frac{\pi}{3}$, $x = \frac{n_4\pi}{2}$, $x = \frac{2n_5\pi}{5}$

$\Rightarrow$ $x = 0, 2\pi$ are common solutions.

(C) $2^{\frac{1}{1-|\cos x|}} = 4$

$\Rightarrow \frac{1}{1 - |\cos x|} = 2 \Rightarrow 1 - |\cos x| = \frac{1}{2}$

$\Rightarrow |\cos x| = \frac{1}{2} \Rightarrow \cos x = \pm\frac{1}{2}$

$\therefore$ In $(-\pi, \pi)$ there are 4 solutions

(D) $\tan\theta + \tan2\theta + \tan3\theta = \tan\theta \tan2\theta \tan3\theta$

$\Rightarrow \theta + 2\theta + 3\theta = n\pi \Rightarrow \theta = n\pi/6$

$\Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ satisfy equation only.

Comprehension # 1

$(1 + a)\cos\theta \cos(2\theta - b) = (1 + a\cos2\theta)\cos(\theta - b)$

$\Rightarrow \cos\theta\cos(2\theta - b) + a\cos\theta\cos(2\theta - b)$

$= \cos(\theta - b) + a\cos2\theta\cos(\theta - b)$

$\Rightarrow 2\cos\theta\cos(2\theta - b) + 2a\cos\theta\cos(2\theta - b)$

$= 2\cos(\theta - b) + 2a\cos2\theta\cos(\theta - b)$

$\Rightarrow \cos(3\theta - b) + \cos(\theta - b) + a(\cos(3\theta - b) + \cos(\theta - b))$

$= 2\cos(\theta - b) + a(\cos(3\theta - b) + \cos(\theta + b))$

$\Rightarrow \cos(3\theta - b) + a\cos(\theta - b) = \cos(\theta - b) + a\cos(\theta + b)$

$\Rightarrow \cos(3\theta - b) - \cos(\theta - b) = a(\cos(\theta + b) - \cos(\theta - b))$

$\Rightarrow 2\sin(2\theta - b)\sin\theta = 2a\sin\theta\sin b$

$\Rightarrow \sin\theta(\sin(2\theta - b) - a\sin b) = 0$

$\Rightarrow \sin\theta = 0$ or $\sin(2\theta - b) = a\sin b$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z.$

and $\sin(2\theta - b) = a\sin b.$

$\Rightarrow \sin(2\theta - b) = \sin(\sin^{-1}(a\sin b))$

$\Rightarrow 2\theta - b = n\pi + (-1)^n \sin^{-1}(a\sin b)\ n \in Z$

1. $S_1 = n\pi,\ n \in Z.$

$S_2 = \dfrac{1}{2}(n\pi + (-1)^n \sin^{-1}(a\sin b) + b)\ n \in Z.$

2. $|a\sin b| \le 1 \rightarrow$ For S_2 non empty.

3. If $a = 0$

$\sin(2\theta - b) = 0$

$2\theta - b = n\pi\ \ n \in Z$

for S_2 a subset of $(0,\pi)$

$0 < \dfrac{n\pi + b}{2} < \pi \quad n \in Z$

$\Rightarrow -n\pi < b < 2\pi - n\pi.$

$b \in (-n\pi, 2\pi - n\pi),\ n \in Z.$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

1. $\sin A = \sin B$ & $\cos A = \cos B$

$A = n\pi \pm (-1)^n B,\ A = 2n\pi \pm B$

$\therefore$ Common solution is $A = 2n\pi + B$

3. $\dfrac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$

$\Rightarrow \dfrac{3}{4}(1 - \cos^2 x) = (\cos^2 x + \cos x)^2$

$\Rightarrow 3(1 - \cos^2 x) - 4\cos^2 x(1 + \cos x)^2 = 0$

$\Rightarrow (1 + \cos x)[3 - 3\cos x - 4\cos^2 x - 4\cos^3 x] = 0$

$\Rightarrow (1 + \cos x)[-4\cos^3 x - 4\cos^2 x - 3\cos x + 3] = 0$

$\Rightarrow (1 + \cos x)(\cos x - \dfrac{1}{2})(4\cos^2 x + 6\cos x + 6) = 0$

$\Rightarrow \cos x = -1,\ \cos x = \dfrac{1}{2},\ 4\cos^2 x + 6\cos x + 6 = 0$

$x = \pi\ ;\ x = \dfrac{\pi}{3}\ ;\ \dfrac{5\pi}{3}\ ;\ D < 0$

But $\dfrac{5\pi}{3}$ does not satisfy the equation

$\therefore\ x = 2n\pi + \dfrac{\pi}{3}$

or $x = 2n\pi \pm \pi,\ n \in I$

10. $(1 - \tan\theta)(1 + \sin2\theta) = 1 + \tan\theta$

$(1 - \tan\theta)(1 + \dfrac{2\tan\theta}{1 + \tan^2\theta}) = (1 + \tan\theta)$

$(1 - \tan\theta)(\tan\theta + 1)^2 - (1 + \tan\theta)(1 + \tan^2\theta) = 0$

$(1 + \tan\theta)\{1 - \tan^2\theta - 1 - \tan^2\theta\} = 0$

$(1 + \tan\theta)\tan^2\theta = 0$

$\theta = n\pi$ or $\theta = n\pi - \dfrac{\pi}{4},\ n \in I$

13. $\sin3\alpha = 4\sin\alpha\sin(x + \alpha)\sin(x - \alpha)$

$\Rightarrow \sin3\alpha = 2\sin\alpha\{\cos2\alpha - \cos2x\}$

$\Rightarrow \sin3\alpha = \sin3\alpha - \sin\alpha - 2\sin\alpha\cos2x$

$\Rightarrow \sin\alpha(2\cos2x + 1) = 0$

$\therefore\ \alpha \ne n\pi \quad \cos2x = -\dfrac{1}{2}$

$2x = 2n\pi \pm \dfrac{2\pi}{3},\ x = n\pi \pm \dfrac{\pi}{3}$

14. $\sin3x < \sin x$

$\Rightarrow \sin3x - \sin x < 0$

$\Rightarrow 3\sin x - 4\sin^3 x - \sin x < 0$

$\Rightarrow 4\sin^3 x - 2\sin x > 0$

$\Rightarrow 2\sin x(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) > 0$

$-\ \ +\ \ -\ \ +$

$-1/\sqrt{2}\quad 0\quad 1/\sqrt{2}$

$\Rightarrow -\dfrac{1}{\sqrt{2}} < \sin x < 0$ or $\sin x > \dfrac{1}{\sqrt{2}}$

$2n\pi - \dfrac{\pi}{4} < x < 0 + 2n\pi,\ 2n\pi + \pi < x < 2n\pi + \dfrac{5\pi}{4}$

and $2n\pi + \dfrac{\pi}{4} < x < \dfrac{3\pi}{4} + 2n\pi$

16. Given $x - y = \dfrac{\pi}{4}$ ...(i)

$\cot x + \cot y = 2$ ....(ii)

From(ii), $\sin(x + y) = 2\sin x.\sin y$

$= \cos(x - y) - \cos(x + y)$

$= \cos\dfrac{\pi}{4} - \cos(x + y)$

$$\Rightarrow \quad \sin(x+y)+\cos(x+y)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin(x+y)+\frac{1}{\sqrt{2}}\cos(x+y)=\frac{1}{2}$$

$$\Rightarrow \quad \cos(x+y-\frac{\pi}{4})=\cos\frac{\pi}{3}$$

$$\Rightarrow \quad x+y-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$$

$$\Rightarrow \quad x+y=2n\pi\pm\frac{\pi}{3}+\frac{\pi}{4} \qquad \ldots\text{(iii)}$$

from $n = 0$, $x + y = \frac{7\pi}{12}$ (since $x,y > 0$) ...(iv)

From (i) and (iv), $x=\frac{5\pi}{12}, y=\frac{\pi}{6}$

Hence least positive values of x and y are

$\frac{5\pi}{12}$ and $\frac{\pi}{6}$ respectively.

17. $\sin x + \cos(k + x) + \cos(k - x) = 2$

$2 \cos k. \cos x + \sin x \qquad x = 2$

This equation is of the form $a \cos x + b \sin x = c$

Here $a = 2 \cos k$, $b = 1$ and $c = 2$

Since for real solutions, $|c| \le \sqrt{a^2+b^2}$

$$\therefore \quad |2| \le \sqrt{1+4\cos^2 k} \Rightarrow 2 \le \sqrt{1+4\cos^2 k}$$

$$\Rightarrow \quad \cos^2 k \ge \frac{3}{4} \qquad \Rightarrow \quad \sin^2 k \le \frac{1}{4}$$

$$\Rightarrow \quad \sin^2 k-\frac{1}{4}\le 0 \Rightarrow \left(\sin k+\frac{1}{2}\right)\left(\sin k-\frac{1}{2}\right)\le 0$$

$$\Rightarrow \quad -\frac{1}{2}\le \sin k \le \frac{1}{2} \quad \Rightarrow \quad n\pi-\frac{\pi}{6}\le k \le n\pi+\frac{\pi}{6}$$

EXERCISE - 04[B] BRAIN STORMING SUBJECTIVE EXERCISE

2. $\sqrt{16\cos^4 x-8\cos^2 x+1}+\sqrt{16\cos^4 x-24\cos^2 x+9}$

$= 2$

$$\sqrt{(4\cos^2 x-1)^2} + \sqrt{(4\cos^2 x-3)^2} = 2$$

$|4\cos^2x - 1| + |4\cos^2x - 3| = 2$

Case-I : If $\cos^2x < \frac{1}{4}$

then $1 - 4\cos^2x + 3 - 4\cos^2x = 2$

$\Rightarrow 8\cos^2x = 2$

$\cos^2x = \frac{1}{4}$ (reject)

Case-II : If $\frac{1}{4} \le \cos^2x \le \frac{3}{4}$ then

$4\cos^2x - 1 + 3 - 4\cos^2x = 2$

$2 = 2$ (identity)

$$\cos x \in \left[\frac{-\sqrt{3}}{2},-\frac{1}{2}\right] \cup \left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow \quad x \in \left[n\pi+\frac{\pi}{6}, n\pi+\frac{\pi}{3}\right]\cup\left[n\pi+\frac{2\pi}{3}, n\pi+\frac{5\pi}{6}\right] n \in I$$

Case III : If $\cos^2x > \frac{3}{4}$ then

$4\cos^2x - 1 + 4\cos^2x - 3 = 2$

$\cos^2x = \frac{3}{4}$ (rej.)

4. $2\sin(3x + \frac{\pi}{4}) = \sqrt{1+8\sin 2x\cos^2 2x}$

$$\Rightarrow 2\sin\left(3x+\frac{\pi}{4}\right) = \sqrt{1+4\sin 4x\cos 2x}$$

$$\Rightarrow 4\sin^2\left(3x+\frac{\pi}{4}\right) = 1 + 4\sin4x \cos2x$$

$$\Rightarrow 2 - 2\cos\left(6x+\frac{\pi}{2}\right) = 1 + 2\sin6x + 2\sin2x$$

$$\Rightarrow \sin2x = \frac{1}{2} \Rightarrow x = \frac{n\pi}{2}+(-1)^n\frac{\pi}{12},\ n \in I$$

$$\Rightarrow x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

$\because \sin\left(3x+\frac{\pi}{4}\right)\ge 0$ so $n = \frac{\pi}{12}, \frac{17\pi}{12}$ satisfies

so general solution is $x = 2n\pi + \frac{\pi}{12}$, $2n\pi + \frac{17\pi}{12}$

7. $\sin^4x + \cos^4x + \sin2x + a = 0$

$\Rightarrow 1 - \frac{1}{2}\sin^2 2x + \sin2x + a = 0$

Now $a = \frac{1}{2}\sin^2 2x - \sin2x - 1$

or $\sin2x = 1 \pm \sqrt{2a+3}$

$\Rightarrow 2a + 3 = (\sin2x - 1)^2$ or $\sin2x = 1 - \sqrt{2a+3}$

$\Rightarrow 0 \le (2a + 3) \le 4$

or $2x = n\pi + (-1)^n \sin^{-1}(1 - \sqrt{2a+3})$

$\Rightarrow -\frac{3}{2} \le a \le \frac{1}{2}$

or $x = \frac{1}{2}[n\pi + (-1)^n \sin^{-1}(1 - \sqrt{2a+3})$

8. Let $\pi 3^x = \theta$

$\cos\theta - 2\cos^2\theta + 2\cos 4\theta - \cos 7\theta$

$= \sin\theta + 2\sin^2\theta - 2\sin 4\theta + 2\sin 3\theta - \sin 7\theta$

$\Rightarrow 2\sin 4\theta \sin 3\theta + 2\cos 4\theta - 2$

$= -2\sin 3\theta \cos 4\theta - 2\sin 4\theta + 2\sin 3\theta$

$\Rightarrow \sin 3\theta(\sin 4\theta + \cos 4\theta - 1) + (\cos 4\theta + \sin 4\theta - 1)$

$= 0$

$\Rightarrow (\sin 3\theta + 1)(\sin 4\theta + \cos 4\theta - 1) = 0$

$\Rightarrow \sin 4\theta + \cos 4\theta = 1$ or $\sin 3\theta = -1$

$\cos\left(\frac{\pi}{4} - 4\theta\right) = \frac{1}{\sqrt{2}}$ $\quad 3\theta = 2k\pi - \frac{\pi}{2}$

$\Rightarrow 4\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow \theta = \frac{m\pi}{2} + \frac{\pi}{8}, \ \theta = \frac{n\pi}{2}, \ \theta = \frac{2k\pi}{3} - \frac{\pi}{6}$

$\pi 3^x = \frac{m\pi}{2} + \frac{\pi}{8}, \ \pi 3^x = \frac{n\pi}{2}, \ \pi 3^x = \frac{2k\pi}{3} - \frac{\pi}{6}$

$x = \log_3\left(\frac{1}{8} + \frac{m}{2}\right), \ x = \log_3\left(\frac{n}{2}\right), \ x = \left(\frac{2k}{3} - \frac{1}{6}\right)$

$m \in N \cup \{0\}, \ n \in N \quad K \in N$

10. Second equation reduce to

$3\sin x \cos y - \sin y \cos x = 0$

or $\cos x \sin y = \frac{3}{4}$ (using first equation)

Now we have

$\sin x \cos y = \frac{1}{4}$ ....(i)

& $\cos x \sin y = \frac{3}{4}$ ....(ii)

By adding & subtracting (i) & (ii) we get

$\sin(x + y) = 1$ & $\sin(x - y) = -\frac{1}{2}$

$x + y = (4K + 1)\frac{\pi}{2}$

$x - y = 2m\pi - \frac{\pi}{6}, \ 2m\pi - \frac{5\pi}{6}$

Now consides

$x + y = (4K + 1)\frac{\pi}{2}$ & $x - y = 2m\pi - \frac{\pi}{6}$

$\Rightarrow x = (4K+1)\frac{\pi}{4} + m\pi - \frac{\pi}{12}$

& $y = (4K+1)\frac{\pi}{4} - m\pi + \frac{\pi}{12}$

Now consider

$x + y = (4K + 1)\frac{\pi}{2}$ & $x - y = 2m\pi - \frac{5\pi}{6}$

$\Rightarrow x = (4K+1)\frac{\pi}{4} + m\pi - \frac{5\pi}{12}$

& $y = (4K+1)\frac{\pi}{4} - m\pi + \frac{5\pi}{12}$

4. $0 < x < \pi$ and $\cos x + \sin x = 1/2$

$2(\cos x + \sin x) = 1$

$\Rightarrow 4[\cos^2 x + \sin^2 x + 2\sin x\cos x] = 1$

$\Rightarrow 4[1 + \tan^2 x + 2\tan x] = \sec^2 x$

$\Rightarrow 4[1 + \tan^2 x + 2\tan x] = 1 + \tan^2 x$

$\Rightarrow 4 + 4\tan^2 x + 8\tan x = 1 + \tan^2 x$

$\Rightarrow 3\tan^2 x + 8\tan x + 3 = 0$

$$\tan x = \frac{-8 \pm \sqrt{64 - 4.3.3}}{2.3} = \frac{-8 \pm 2\sqrt{7}}{2.3} = \frac{-4 \pm \sqrt{7}}{3}$$

5. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$2\cos(\beta - \gamma) + 2\cos(\gamma - \alpha) + 2\cos(\alpha - \beta) = -3$

$1 + 1 + 1 + 2(\cos\beta\cos\gamma + \sin\beta\sin\gamma)$

$+ 2(\cos\gamma\cos\alpha + \sin\gamma\sin\alpha)$

$+ 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 0$

$(\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + (\sin^2\gamma + \cos^2\gamma)$

$+ 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha$

$+ 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha = 0$

$(\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma$

$+ 2\sin\gamma\sin\alpha) + (\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$+ 2\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha) = 0$

$(\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$

Only Possible when

$\sin\alpha + \sin\beta + \sin\gamma = 0$

$\cos\alpha + \cos\beta + \cos\gamma = 0$

6. $\sin\theta + \sin 4\theta + \sin 7\theta = 0$

$$2\sin\left(\frac{\theta + 7\theta}{2}\right)\cos\left(\frac{7\theta - \theta}{2}\right) + \sin 4\theta = 0$$

$\Rightarrow \sin 4\theta[2\cos 3\theta + 1] = 0$

$\Rightarrow \sin 4\theta = 0 \Rightarrow 4\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\Rightarrow \theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

but 0 and π are not included.

and $2\cos 3\theta + 1 = 0 \Rightarrow \cos 3\theta = \frac{-1}{2}$

$\Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$

$\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}$

but $\frac{10\pi}{9} \notin (0, \pi)$

So, $\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

5. $\sin 2\theta = \cos 4\theta$

$\sin 2\theta = 1 - 2\sin^2 2\theta$

$2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow \sin 2\theta = -1, \frac{1}{2}$

$\therefore \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

Now $\tan\theta = \cot 5\theta$ (Given)

All three obtained values of θ satisy the given equation.

Hence, number of values of θ are 3.

6. $$\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$$

$$\Rightarrow \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow 2\cos\frac{2\pi}{n}\sin\frac{2\pi}{n} = \sin\frac{3\pi}{n}$$

$$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \frac{4\pi}{n} = K\pi + (-1)^K \frac{3\pi}{n}$$

If $K = 2m \quad \Rightarrow \quad \frac{\pi}{n} = 2m\pi$

$\Rightarrow \quad n = \frac{1}{2m} \quad \Rightarrow \quad n = \frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots\ldots..$

If $K = 2m + 1 \quad \Rightarrow \quad \frac{7\pi}{n} = (2m+1)\pi$

$\Rightarrow \quad n = \frac{7}{2m+1} \quad \Rightarrow \quad n = 7, \frac{7}{3}, \frac{7}{5} \ldots\ldots.$

Possible value of n is 7

7. $\tan(2\pi - \theta) > 0$

$\Rightarrow$ $2\pi - \theta$ lies in I on III quadrant

$\Rightarrow$ $\because$ lies in II or IV Quadrant ....(i)

$\because \quad -1 < \sin\theta < -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta \in \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)$ ......(ii)

$\therefore$ By (i) and (ii) : $\theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

Also, given $2\cos\theta(1 - \sin\phi)$

$$= \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1$$

$$\Rightarrow 2\cos\theta - 2\cos\theta\sin\phi = \frac{\sin^2\theta\cos\phi}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}} - 1$$

$\Rightarrow \quad 2\cos\theta - 2\cos\theta\sin\phi = 2\sin\theta\cos\phi - 1$

$$\Rightarrow \sin(\theta + \phi) = \frac{1 + 2\cos\theta}{2}$$

$$\Rightarrow \frac{1}{2} < \sin(\theta + \phi) < 1$$

$$\Rightarrow \frac{\pi}{6} < \theta + \phi < \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} < \theta + \phi < \frac{17\pi}{6}$$

$$\because \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right) \Rightarrow \frac{13\pi}{6} < \theta + \phi < \frac{17\pi}{6}$$

$$\Rightarrow \frac{\pi}{2} < \phi < \frac{4\pi}{3}$$