EXERCISE - 01 CHECK YOUR GRASP

4. $f(x) = x^3 + 6x^2 + (9 + 2k)x + 1$

$f'(x) = 3x^2 + 12x + (9 + 2k)$

$\Rightarrow 3x^2 + 12x + (9 + 2k) \geq 0 \ \forall \ x \in R \quad \Rightarrow D \leq 0$

$\Rightarrow 12.12 - 12\,(9 + 2k) \leq 0$

$3 - 2k \leq 0 \quad \Rightarrow \quad k \geq \frac{3}{2}.$

6. $f(x) = \sin x - \cos x - ax + b$

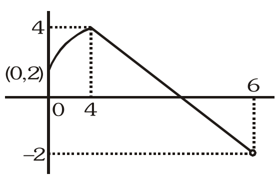
$f'(x) = \cos x + \sin x - a \leq 0 \ \forall \ x \in R$

$\Rightarrow \quad a \geq \cos x + \sin x \ \forall \ x \in R$

$\Rightarrow \quad a \geq \sqrt{2}$

9. $$f(x) = \begin{cases} \dfrac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})} & , \quad 0 < x < 4 \\ 4 & , \quad x = 4 \\ 16 - 3x & , \quad 4 < x < 6 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 2 + \sqrt{x} & , \quad 0 < x < 4 \\ 4 & , \quad x = 4 \\ 16 - 3x & , \quad 4 < x < 6 \end{cases}$$

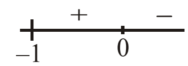


So f(x) is continuous only

11. $\ell n(1 + x) - x \leq 0$

Let $f(x) = \ell n\,(1 + x) - x$, {Domain is $(-1, \infty)$}

$f'(x) = \frac{1}{x+1} - 1 = \frac{-x}{x+1}$



$\Rightarrow f(x) \leq f(0) \ \forall \ x \in (-1, \infty) \Rightarrow f(x) \leq 0 \ \forall \ x \in (-1, \infty).$

13. $f(x) = 3\tan x + x^3 - 2$, $f'(x) = 3\,(\sec^2 x + x^2) > 0$

$\Rightarrow$ f(x) is increasing in $\forall \ x \in (0, \pi/4)$

$f(0) < 0 \ \& \ f\left(\frac{\pi}{4}\right) > 0$

$\Rightarrow$ f(x) =0 has exactly one root in $\left(0, \frac{\pi}{4}\right)$.

15. $f(-2) = f(3) = 0$

f(x) is continuous in $[-2, 3]$ & derivable in $(-2, 3)$ so Rolle's theorem is applicable.

so $\exists$ $c \in (-2, 3)$ such that $f'(c) = 0$

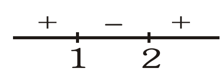
$\Rightarrow \quad \frac{2c^3 - 5c^2 + 4c - 1}{(c-1)^2} = 0 \quad \Rightarrow \quad c = 1/2$

18. Using LMVT in [2, 4]

$f'(c) = \frac{f(4) - f(2)}{4 - 2} = \frac{f(4) + 4}{2}$

$f'(x) \geq 6 \quad \Rightarrow \quad \frac{f(4)+4}{2} \geq 6 \quad \Rightarrow \quad f(4) \geq 8$

22. $f'(x) = e^x(x - 1)(x - 2)$



f(x) is increasing in $(-\infty, -2)$ & $(-2, -1)$ & $(2, \infty)$

EXERCISE - 02 BRAIN TEASERS

2. $y = \frac{2(x-2)+3}{x-2}$

$y = 2 + \frac{3}{(x-2)}$

$\frac{dy}{dx} = \frac{-3}{(x-2)^2} < 0$

$\therefore \quad$ y decreases $\forall \ x \in R$

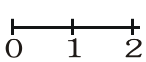
Now, $x = \frac{2y-1}{y-2}$

$xy - 2x = 2y - 1$

$y(x - 2) = 2x - 1$

$y = \frac{2x-1}{x-2} = f^{-1}(x)$ [Also, $y \in R - \{1\}$]

3.



$\because$ f(0) = f(1) & 'f' is continuous in [0, 1] & derivable in (0, 1)

$\therefore$ $f'(c_1) = 0$ for atleast one $c_1 \in (0, 1)$

similarly, $\because$ f(1) = f(2)

$\therefore$ $f'(c_2) = 0$ for atleast one $c_2 \in (1, 2)$

$\Rightarrow f'(c_1) = f'(c_2)$

$\Rightarrow f''(c) = 0$ for atleast one $c \in (c_1, c_2)$

4. f(x) is continuous in [0, 1] & derivable in (0, 1)

Consider the interval $\left[0, \frac{1}{n}\right]$ where $n \in I^+$

$f(0) = f\left(\frac{1}{n}\right)$

$\Rightarrow f'(c) = 0$ for atleast one $c \in \left(0, \frac{1}{n}\right)$

we can have such infinite number of points.

5. $\phi(x) = f^3(x) - 3f^2(x) + 4f(x) + 5x + 3\sin x + 4\cos x$

$\phi'(x) = (3f^2(x) - 6f(x) + 4)f'(x) + 5 + 3\cos x - 4\sin x$ ..(i)

$3\cos x - 4\sin x \geq -5$

$5 + (3\cos x - 4\sin x) \geq 0$

also $3f^2(x) - 6f(x) + 4 > 0 \quad \because D < 0$

$\phi'(x) > 0 \quad \forall f'(x) > 0$

Now let $f'(x) = -11$

$\phi'(x) = -11(3f^2(x) - 6f(x) + 4) + 5 + 3\cos x - 4\sin x$

Now $3f^2(x) - 6f(x) + 4 \geq 1$

$\Rightarrow -11(3f^2(x) - 6f(x) + 4) \leq -11$ -- (ii)

$3\cos x - 4\sin x \leq 5$

$\Rightarrow 5 + (3\cos x - 4\sin x) \leq 10$ -- (iii)

(ii)+(iii)

$\Rightarrow -11(3f^2(x)-6f(x)+4)+5+(3\cos x-4\sin x)\leq -1$

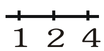
$\Rightarrow \phi'(x) \leq -1$

8. $\dfrac{dx}{dt} = \dfrac{-2t}{(1+t^2)^2}, \quad \dfrac{dy}{dt} = \dfrac{-(1+3t^2)}{t^2(1+t^2)^2}$

$\dfrac{dy}{dx} = \dfrac{1+3t^2}{2t^3} \Rightarrow \dfrac{dy}{dx} > 0 \ \forall t > 0$

Now, $x = \dfrac{1}{1+t^2}$

$t > 0 \Rightarrow x \in (0, 1)$

9. 

Applying LMVT in [1, 2]

$\dfrac{f(2)-f(1)}{2-1} = f'(c_1) \ \forall c_1 \in (1, 2)$

$f(2) - 2 \leq 2 \ \{\because f'(x) \leq 2\} \Rightarrow f(2) \leq 4$ ... (1)

Similarly applying LMVT in [2, 4]

$\dfrac{f(4)-f(2)}{4-2} = f'(c_2) \ \forall c_2 \in (2, 4)$

$\dfrac{8-f(2)}{2} \leq 2 \Rightarrow f(2) \geq 4$ ... (2)

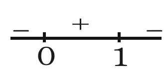
from (1) & (2) $\quad f(2) = 4$

11. $3x^2 - 2x^3 = \log_2\left(x + \dfrac{1}{x}\right), \ x > 0$

$f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$= 6x(1 - x)$



$f(x) \leq f(1)$

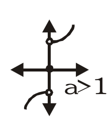
$f(x) \leq 1$

$\Rightarrow$ LHS ≤ 1 & RHS ≥ 1

LHS = RHS = 1 for $x = 1$

$\therefore$ equation has exactly one solution

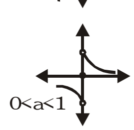
13. $(\ell n\, a)\, h(x) = \ell n\, a^{\{a^{|x|}\operatorname{sgn} x\}} + \ell n\, a^{[a^{|x|}\operatorname{sgn} x]}$



$(\ell n\, a)\, h(x) = \left(\{a^{|x|}\operatorname{sgn} x\} + [a^{|x|}\operatorname{sgn} x]\right)\ell n\, a$

$(\ell n\, a)\, h(x) = (a^{|x|}\operatorname{sgn} x)(\ell n\, a)$

$\Rightarrow h(x) = a^{|x|}\operatorname{sgn}(x)$



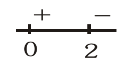
If $a > 1 \Rightarrow$ 'h' is odd & increasing

$0 < a < 1 \Rightarrow$ 'h' is odd but neither increasing nor decreasing.

14. $x^2 e^{2-|x|} - 1 = 0$

$f(x) = \dfrac{e^2x^2}{e^x} - 1 \ \forall x \geq 0$

$f'(x) = e^2 \dfrac{\{2xe^x - x^2e^x\}}{e^{2x}} = \dfrac{e^2.x(2-x)}{e^x}$



$\Rightarrow$ f increases in $(0, 2)$, f decreases in $(2, \infty)$

Also $f(0) < 0$ & $f(2) > 0 \Rightarrow$ Exactly one root in $(0, 2)$

$\lim\limits_{x\to\infty} f(x) < 0$

$\Rightarrow$ exactly one root in $(2, \infty)$

$\Rightarrow$ exactly 2 roots in $(0, \infty)$

$\Rightarrow$ equation has exactly 4 roots

$\because f(x)$ is even function.

15. $f(x) = \dfrac{1}{(x+1)^3} - 3x + \sin x$

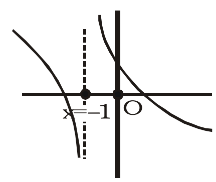
Domain of 'f' is $(-\infty, -1) \cup (-1, \infty)$

$f'(x) = -3\left(\dfrac{1}{(x+1)^4} + 1\right) + \cos x.$

$\Rightarrow f'(x) < 0 \Rightarrow$ f is decreasing

$\lim\limits_{x\to -1^+} f(x) \to \infty \qquad \lim\limits_{x\to -1^-} f(x) \to -\infty$

$\lim\limits_{x\to\infty} f(x) \to -\infty \quad \lim\limits_{x\to -\infty} f(x) \to \infty$



$\Rightarrow f(x) = 0$ has exactly two roots.

16. $f(x) = \left(\dfrac{\sqrt{p+4}}{1-p} - 1\right)x^5 - 3x + \ell n\, 5$

$f'(x) = \left(\dfrac{\sqrt{p+4}}{1-p} - 1\right)5x^4 - 3 < 0 \ \forall x \in R$

$\Rightarrow \quad \dfrac{\sqrt{p+4}}{1-p} - 1 \leq 0$

If $-4 \leq p < 1$ then

$\Rightarrow \quad \sqrt{p+4} \leq 1 - p \quad \Rightarrow \quad p + 4 \leq 1 - 2p + p^2$

$\Rightarrow \quad p^2 - 3p - 3 \geq 0$

$\Rightarrow \quad p \leq \dfrac{3-\sqrt{21}}{2}$ or $\dfrac{3+\sqrt{21}}{2} \leq p$

$\Rightarrow \quad p \in \left[-4, \dfrac{3-\sqrt{21}}{2}\right]$

If $p > 1$ then $\sqrt{p+4} \geq 1 - p$

$\Rightarrow$ Always true for $p > 1$

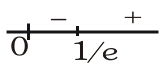
$\Rightarrow p \in \left[-4, \dfrac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

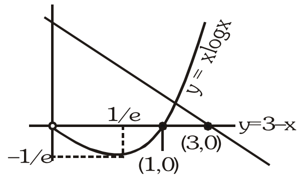
Match the Column :

1. **(A)** $x \log x = 3 - x$

$y = x \log x$

$y' = 1 + \log x$





$$\lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0$$

$$\lim_{x\to\infty} x\log x \to \infty$$

There is exactly one root of the equation in (1, 3).

(B) Let $g(x) = \int(4ax^3 + 3bx^2 + 2cx + d)\,dx$

$\Rightarrow \quad g(x) = ax^4 + bx^3 + cx^2 + dx + K$

$\Rightarrow \quad g(0) = g(3) = K$

$\{\because \quad 27a + 9b + 3c + d = 0\}$

$\therefore$ By Rolle's Theorem $g'(x) = 0$ has atleast one root in (0, 3).

(C) Let the required inteval be (a, b).

By LMVT

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \quad \frac{b+\frac{1}{b}-a-\frac{1}{a}}{b-a} = 1 - \frac{1}{c^2}$$

$$\Rightarrow \quad 1 - \frac{1}{ab} = 1 - \frac{1}{3} \Rightarrow ab = 3$$

(D) $$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \quad \frac{(2b-b^2)-(2a-a^2)}{b-a} = 2(1-c)$$

$$\Rightarrow \quad \frac{2(b-a)-(b^2-a^2)}{b-a} = 1$$

$\Rightarrow 2 - (b + a) = 1 \Rightarrow (b + a) = 1$

Assertion and Reason :

1. Statement-II :

$\because$ f(x) is continuous, derivable & $f(1) = f(2) = 0$

$\Rightarrow$ $f'(x) = 0$ has atleast one root in (1, 2).

$\Rightarrow$ $e^{10x}(2x - 3) + 10\, e^{10x}(x^2 - 3x + 2) = 0$

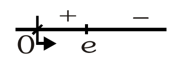
has atleast one root in (1, 2).

$\Rightarrow$ $10x^2 - 28x + 17 = 0$ has at least one root in (1, 2).

Statement-I is true & statement-II explains statement-I.

3. Consider $f(x) = x^{1/x}$

$$f'(x) = x^{1/x}\left(\frac{1-\ell n\, x}{x^2}\right) \quad \forall \quad x > 0$$



$\therefore$ at $x = e$, f(x) has absolute maximum value.

$3^{1/3} > 4^{1/4} = 2^{1/2}$.

Hence both statements are true & statement-II explains statements I.

Comprehension # 1 :

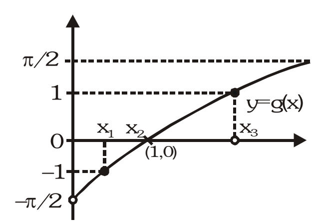
$f(x) = \tan^{-1}(\ell n\, x)$

1. $\because$ $\tan^{-1}(x)$ & $\ell n\, x$ are increasing functions.

$\Rightarrow$ f(x) is also increasing function.

2. $$\lim_{x\to 0^+} \tan^{-1}(\ell n\, x) \to -\frac{\pi}{2}$$

$$\lim_{x\to\infty} \tan^{-1}(\ell n\, x) \to \frac{\pi}{2} \quad \Rightarrow \quad \text{range of 'f' is } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$



3. From graph, g(x) is discontinuous at $x = x_1, x_2, x_3$

$\tan^{-1}(\ell n\, x_1) = -1$; $\tan^{-1}(\ell n\, x_2) = 0$; $\tan^{-1}(\ell n\, x_3) = 1$

$$\Rightarrow \quad x_1 = \frac{1}{e^{\tan 1}}; \quad x_2 = 1; \quad x_3 = e^{\tan 1}$$

$$x_1 + x_2 + x_3 = e^{\tan 1} + \frac{1}{e^{\tan 1}} + 1 > 3.$$

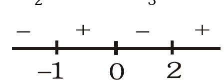
Comprehension # 2 :

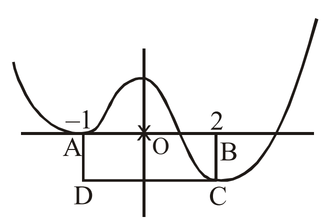
$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x$

$\quad = 12x(x - 2)(x + 1)$

$\therefore \quad a_1 = -1,\ a_2 = 0\ \&\ a_3 = 2.$





on the basis of above graph, the given questions can be solved.

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

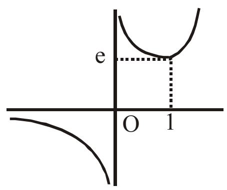
1. (b) $f(x) = \dfrac{e^x}{x}$ {Domain of 'f' is R – {0}}

$f'(x) = \dfrac{x.e^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2}$

$\underset{0}{-}\quad - \quad \underset{1}{+}$

$\lim\limits_{x\to 0^+} \dfrac{e^x}{x} \to \infty \qquad \lim\limits_{x\to \infty} \dfrac{e^x}{x} \to \infty$

$\lim\limits_{x\to 0^-} \dfrac{e^x}{x} \to -\infty \qquad \lim\limits_{x\to -\infty} \dfrac{e^x}{x} = 0$



2. $1 - f(x) - f^3(x) > f(1 - 5x)$

$\Rightarrow$ $f'(x) = -1 - 3x^2 < 0 \Rightarrow$ f is decreasing

Now $f(f(x)) > f(1 - 5x)$

$\Rightarrow f(x) < 1 - 5x$ $\{\because$ f is decreasing $\}$

$1 - x - x^3 < 1 - 5x.$

8. For a = 1

$f(x) = 2x + 1$ $\Rightarrow$ f is monotonic increasing

If $a \neq 1$

$f'(x) = (a^2 - 1)x^2 + 2(a - 1)x + 2$

$f'(x) \geq 0$ ($\because$ 'f' is monotonic increasing.)

$\Rightarrow D \leq 0$ & $a^2 - 1 > 0$

$4(a - 1)^2 - 8(a - 1)(a + 1) \leq 0$

$(a - 1)\{a - 1 - 2a - 2\} \leq 0 \Rightarrow (a - 1)(-a - 3) \leq 0$

$\Rightarrow (a - 1)(a + 3) \geq 0$ $\Rightarrow a \in (-\infty, -3] \cup [1, \infty)$.

9. $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$

$f'(x) = 2\cos 2x - 8(a + 1)\cos x + (4a^2 + 8a - 14)$

$f'(x) = 2(2\cos^2 x - 1) - 8(a + 1)\cos x + 4a^2 + 8a - 14$

$= 4\{\cos^2 x - 2(a + 1)\cos x\} + 4a^2 + 8a - 16$

$= 4\{\cos x - (a + 1)\}^2 - 20 > 0$

$= \{\cos x - (a + 1)\}^2 - (\sqrt{5})^2 > 0$

$f'(x) = \{\cos x - (a + 1) - \sqrt{5}\}\{\cos x - (a + 1) + \sqrt{5}\} > 0$

$\Rightarrow \cos x > a + 1 + \sqrt{5}$ or $\cos x < (a + 1) - \sqrt{5}$ $\forall x \in R$

$a + 1 + \sqrt{5} < -1$ or $(a + 1) - \sqrt{5} > 1$

$a < -2 - \sqrt{5}$ or $a > \sqrt{5}$

$a \in (-\infty, -2 - \sqrt{5}) \cup (\sqrt{5}, \infty)$

11. $x^2 - 1 > 2x\,\ell n\,x > 4(x - 1) - 2\,\ell n\,x,\ x > 1$

(a) Consider $f(x) = x^2 - 1 - 2x\,\ell n\,x$

$f'(x) = 2\{x - 1 - \ell n\,x\}$

$f''(x) = 2\left\{1 - \dfrac{1}{x}\right\}$

$f''(x) > 0\ \forall\ x > 1$

$\Rightarrow$ f'(x) is increasing $\forall\ x > 1$

$\Rightarrow$ $f'(x) > f'(1)$ $\Rightarrow$ $f'(x) > 0\ \forall\ x > 1$

$\Rightarrow$ f(x) is increasing $\forall\ x > 1$

$\Rightarrow$ $f(x) > f(1)$ $\Rightarrow$ $f(x) > 0\ \forall\ x > 1$

(b) Consider,

$g(x) = 2x\,\ell n\,x + 2\ell n\,x - 4(x - 1)$

$g'(x) = 2(1 + \ell n\,x) + \dfrac{2}{x} - 4$

$g''(x) = \dfrac{2}{x} - \dfrac{2}{x^2}$

$g''(x) = \dfrac{2(x-1)}{x^2} > 0 \quad \forall\ x > 1$

$\Rightarrow$ g'(x) is increasing $\forall\ x > 1$

$\Rightarrow$ $g'(x) > g'(1)$ $\Rightarrow g'(x) > 0\ \forall\ x > 1$

$\Rightarrow$ g(x) is increasing $\forall\ x > 1$

$\Rightarrow$ $g(x) > g(1)\ \forall\ x > 1$

$\Rightarrow$ $g(x) > 0\ \forall\ x > 1.$

15. $f(x) = (x - a)^m (x - b)^n$

$\because$ 'f' is continuous & derivable in [a, b].

& $f(a) = f(b)$

$\therefore$ according to Rolle's theorem,

there must be atleast one root of the equation

$f'(x) = 0$ in (a, b)

consider $f'(x) = 0$

$m(x - a)^{m-1}(x - b)^n + n(x - b)^{n-1}(x - a)^m = 0$

$(x - a)^{m-1}(x - b)^{n-1}\{mx - mb + nx - na\} = 0$

$\Rightarrow$ $x = \dfrac{mb + na}{m + n} \in (a, b).$

20. Consider $g(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}$

Apply LMVT in g(x) in [a, b]

25. (i) $n = 2m$ (even)

$f(x) = x^{2m} + px + q$

$f'(x) = 2mx^{2m-1} + p = 0$

$\Rightarrow$ f'(x) can have exactly one point of local minima or maxima.

$\Rightarrow$ f(x) can not have more than two real roots.

(ii) $n = 2m - 1$ (odd)

$f(x) = x^{2m-1} + px + q$

$f'(x) = (2m-1)x^{2m-2} + p$

If $p > 0 \Rightarrow$ no real root of f'(x)

$p < 0 \Rightarrow$ 2 real roots of f'(x)

$\Rightarrow$ f'(x) can have one maxima & one minima.

$\Rightarrow$ f(x) cannot have more than 3 real roots.

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. $a + b = 4$

$b - a = t$ (say)

$\therefore \quad b = \dfrac{4+t}{2}$; $a = \dfrac{4-t}{2}$

Let $h(t) = \displaystyle\int_0^{\frac{4-t}{2}} g(x)dx + \int_0^{\frac{4+t}{2}} g(x)dx$

$h'(t) = \dfrac{1}{2}\left\{g\left(\dfrac{4+t}{2}\right) - g\left(\dfrac{4-t}{2}\right)\right\}$

Now $a < 2$

$\Rightarrow \dfrac{4-t}{2} < 2 \quad \Rightarrow t > 0$

$\Rightarrow \left(\dfrac{4+t}{2}\right) > \left(\dfrac{4-t}{2}\right) \Rightarrow h'(t) > 0$

($\because$ g is an increasing function.)

$\Rightarrow$ h increases as t (i.e. b – a) increases.

2. $f(x) = 8ax - a\sin 6x - 7x - \sin 5x$

$f'(x) = 8a - 6a\cos 6x - 7 - 5\cos 5x > 0 \ \forall x \in R$

$\Rightarrow a > \dfrac{7 + 5\cos 5x}{8 - 6\cos 6x} \ \forall x \in R$

$\Rightarrow$ RHS assumes maximum value for $x = 0$.

$\Rightarrow a > \dfrac{7+5}{8-6} \Rightarrow a > 6$

3. Consider $g(x) = e^{\alpha x}f(x)$

Now $g(a) = g(b) = 0$.

Also 'g' is derivable in [a, b].

$\therefore g'(x) = 0$ for atleast one $x \in (a, b)$ {Rolle's theorm}.

$e^{\alpha x} \cdot f'(x) + \alpha e^{\alpha x}f(x) = 0$

$\Rightarrow f'(x) + \alpha f(x) = 0$

4. By LMVT in $\left[a, \dfrac{a+b}{2}\right]$

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = f'(c_1),\ c_1 \in \left(a, \frac{a+b}{2}\right) \ \ldots\ldots \text{(i)}$$

By LMVT in $\left[\dfrac{a+b}{2}, b\right]$

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = f'(c_2),\ c_2 \in \left(\frac{a+b}{2}, b\right) \ \ldots\ldots \text{(ii)}$$

(i) + (ii) $\Rightarrow f'(c_1) + f'(c_2) = 2$

7. Let $x \in [-2, 4]$

consider the interval $[-2, x]$

By LMVT

$$\frac{f(x) - f(-2)}{x - (-2)} = f'(c),\ c \in (-2, 4)$$

$\Rightarrow -5 \le \dfrac{f(x) - 1}{x+2} \le 5 \ \{\because |f'(x)| \le 5\}$

$\therefore -5x - 10 \le f(x) - 1 \le 5x + 10$

$-5x - 9 \le f(x) \le 5x + 11.$

EXERCISE - 05 [A]   JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

3. Check the option one by one

third option $f(x) = 3x^2 - 2x + 1$

$f'(x) = 6x - 2 \ge 0$ $\quad$ $x \ge 1/3$ it is incorrect

4. $f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1\times(\cos x - \sin x)}{1+(\sin x+\cos x)^2} > 0$$

$\cos x - \sin x > 0$ $\quad$ $\cos x > \sin x$

$\sin x < \cos x$ $\quad$ $\tan x < 1$

$x < \frac{\pi}{4}$ $\quad$ $\boxed{x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)}$

5. $f(x) = \dfrac{1}{e^x + 2e^{-x}}$

$y = \dfrac{1}{e^x + 2e^{-x}}$ $\quad$ Let $e^x = t \in (0, \infty)$

$y = \dfrac{1}{t+\frac{2}{t}} \Rightarrow y = \dfrac{t}{t^2+2} \Rightarrow t^2y - t + 2y = 0$

$D \ge 0$

$1 - 8y^2 \ge 0$

$\Rightarrow 8y^2 - 1 \le 0 \Rightarrow y \in \left[\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

but $y > 0$

$\therefore y \in \left(0, \frac{1}{2\sqrt{2}}\right]$

$\therefore f(0) = \frac{1}{3}$

$\therefore f(c) = \frac{1}{3}$ $(c \in R)$

So Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

EXERCISE - 05 [B]   JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

4. $f(x) = 4x^3 - 3x - p$

$f\left(\frac{1}{2}\right) = -(p+1)$

$f(1) = (1 - p)$

$f(1)\,.\,f\left(\frac{1}{2}\right) = -(1-p^2) \le 0 \quad \because p \in [-1, 1]$

$\therefore$ $f(x) = 0$ has atleast one root in $\left[\frac{1}{2}, 1\right]$

$f'(x) = 3(2x - 1)(2x + 1)$

$\Rightarrow$ $f'(x) > 0 \quad \forall\, x > \frac{1}{2}$

$\Rightarrow$ $f(x) = 0$ has exactly one root in $\left[\frac{1}{2}, 1\right]$

Let the root be $x = \cos\theta$

$\therefore$ $4\cos^3\theta - 3\cos\theta = p$

$\cos 3\theta = p$

$\Rightarrow$ $\theta = \frac{1}{3}\cos^{-1}(p) \Rightarrow x = \cos\left(\frac{1}{3}\cos^{-1}(p)\right)$

6. **(a)** $\cos x - 1 > -\frac{x^2}{2}$ (given) ....(i)

consider $f(x) = \sin(\tan x) - x$

$f'(x) = \cos(\tan x)(1 + \tan^2 x) - 1$

$= (\tan^2 x)\cos(\tan x) + \cos(\tan x) - 1$

$\cos(\tan x) - 1 > -\frac{\tan^2 x}{2}$ from (i)

$(\tan^2 x)\cos(\tan x) + \cos(\tan x) - 1$

$> \tan^2 x \left\{\cos(\tan x) - \frac{1}{2}\right\}$

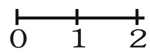
$\Rightarrow$ $f'(x) > \tan^2 x \left\{\cos(\tan x) - \frac{1}{2}\right\}$

$0 \le \tan x \le 1 \quad \left\{\because 0 \le x \le \frac{\pi}{4}\right\}$

$\Rightarrow$ $\cos(\tan x) > \frac{1}{2}$

$\Rightarrow$ $f'(x) > 0 \Rightarrow f(x) \ge f(0) \Rightarrow f(x) \ge 0$

(b) (ii) Consider $g(x) = \int_0^{x^2} f(t)dt$



$g(1) - g(0)$

$= g'(\alpha),\ \alpha \in (0, 1)$ {by LMVT in [0,1]}.......(i)

$g(2) - g(1)$

$= g'(\beta),\ \beta \in (1, 2)$ {by LMVT in [1,2]}.......(ii)

(i) + (ii) $\Rightarrow$ $g(2) - g(0) = g'(\alpha) + g'(\beta)$

$$\Rightarrow \int_0^4 f(t)dt = 2\{\alpha f(\alpha^2) + \beta f(\beta^2)\}$$

8. Let $g(x) = \int p(x)\,dx + K$

$$g(x) = \frac{x^{102}}{2} - 23\,x^{101} - \frac{45x^2}{2} + 1035x + K$$

$$= \frac{x^{102} - 46x^{101} - 45x^2 + 2070x}{2} + K$$

$$= \frac{x(x^{100} - 45)(x - 46)}{2} + K$$

$g(45^{1/100}) = g(46)$

$\Rightarrow$ $g'(x) = 0$ has exactly one root in $(45^{1/100}, 46)$

9. Let $f(x) = \sin x + 2x$ & $g(x) = \dfrac{3x^2 + 3x}{\pi}$

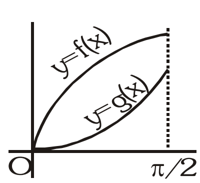
$f'(x) = \cos x + 2$ $\quad g'(x) = \dfrac{6x + 3}{\pi}$

$f''(x) = -\sin x$ $\quad g''(x) = \dfrac{6}{\pi}$

$\Rightarrow$ 'f' is increasing & concave down in $\left[0, \frac{\pi}{2}\right]$

and 'g' is increasing & concave up in $\left[0, \frac{\pi}{2}\right]$

& $f\left(\frac{\pi}{2}\right) > g\left(\frac{\pi}{2}\right)$.



from the graph $f(x) \geq g(x)\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$

11. Consider $g(x) = x^2 - f(x)$

'g' is continuous-derivable

$\therefore$ By Rolle's theorem

$g(1) = g(2) \Rightarrow g'(c_1) = 0$ for atleast one $c_1 \in (1, 2)$

$g(2) = g(3) \Rightarrow g'(c_2) = 0$ for atleast one $c_2 \in (2, 3)$

$g'(c_1) = g'(c_2)$

$\Rightarrow$ $g''(c) = 0$ for atleast one $c \in (c_1, c_2)$.

$\Rightarrow$ $2 - f''(c) = 0$

$\Rightarrow$ $f''(c) = 2$

20. $f(x) = \ell nx + \int_0^x \sqrt{1 + \sin t}\,dt$

$$f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$$

$$f'(x) = \frac{1}{x} + \sqrt{2}\left|\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right|$$

$\because$ $\left|\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right|$ is non-derivable

$\therefore$ $f'(x)$ is non-derivable but continuous.

hence option (A) is incorrect & option (B) is correct.

For option C

$$f(x) = (\ell nx) + \int_0^x \left(\sqrt{1 + \sin x}\right)dx$$

since $f(x)$ is positive increasing function for all $x > 1$

$\Rightarrow$ $|f(x)| = f(x)$ & $|f'(x)| = f'(x)$

Let $f(x) = y$

$$f'(x) - f(x) = \frac{1}{x} - \ell nx + \sqrt{1 + \sin x} - \int_0^x \sqrt{1 + \sin t}\ dt$$

$$f'(x) - f(x) = \frac{1}{x} - \ell nx + \sqrt{1 + \sin x} - \sqrt{2}\int_0^x \left|\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)\right| dt$$

$\dfrac{1}{x} - \ell nx < 0$; when $\alpha > e$

$0 \leq \sqrt{1 + \sin x} \leq \sqrt{2}$.

$$\int_0^x \left|\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)\right| dt > \sqrt{2}\ \ \forall \alpha > \frac{3\pi}{2}$$

$\Rightarrow$ $f'(x) - f(x) < 0\ \ \forall\ \ \alpha > \dfrac{3\pi}{2} > 1$

Hence option (C) is correct.

For option (D) $|f(x)| + |f'(x)| \to \infty$ when $x \to \infty$.

Therefore option (D) is incorrect.

Alternate :

$$f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x} \qquad \text{........(i)}$$

for $x > 1$

$$\frac{1}{x} + \sqrt{1+\sin x} < 1 + \sqrt{2}$$

but $\ell nx + \int_0^x \sqrt{1+\sin t}\,dt$ will always be more than $1+\sqrt{2}$ for some $\alpha > 1$

$\because \quad \int_0^x \sqrt{1+\sin t} > 0$ & ℓnx is increasing in $(1, \infty)$

$\Rightarrow \quad f(x) > f'(x) \ \forall \ \alpha > 1$

$\therefore \quad$ (C) is correct

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$$

$\Rightarrow \quad f'$ is not derivable on $(0, \infty)$

at $\frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore \quad$ (B) is also correct

$f(x)$ is unbounded near $x = 0$ in $(0, 1)$ hence $|f(x)|$ can never be made less than a finite number hence $|f(x)|+|f'(x)|$ can never be less than β.

21. Ans. (A)

$f : (0,1) \to R$

$$f(x) = \frac{b-x}{1-bx} \qquad b \in (0,1)$$

$$\Rightarrow \quad f'(x) = \frac{b^2-1}{(bx-1)^2}$$

$\Rightarrow \quad f'(x) < 0 \ \forall \ x \in (0, 1)$

hence $f(x)$ is decreasing function

hence its range $(-1, b)$

$\Rightarrow \quad$ co-domain $\neq$ range

$\Rightarrow \quad f(x)$ is non-invertible function

22. Ans. 2

Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x$

$f''(x) = 12x^2 - 24x + 24$

$= 12(x^2 - 2x + 2) > 0$

$\Rightarrow \quad f'(x)$ is strictly increasing function

$\because \quad f'(x)$ is cubic polynomial

hence number of roots of $f'(x) = 0$ is 1

$\Rightarrow \quad$ Number of maximum roots of $f(x) = 0$ are 2

Now $f(0) = -1$, $f(1) = 9$, $f(-1) = 15$

$\Rightarrow \quad f(x)$ has exactly 2 distinct real roots.

23. $f(x) = (1 - x)^2 \sin^2 x + x^2$

P : $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow (1 - x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$\Rightarrow (1 - x)^2 \sin^2 x - x^2 + 2x - 2 = 0$

$(1 - x)^2 \cos^2 x + 1 = 0$

which is not possible.

$\therefore$ P is false.

Q : $2f(x) + 1 = 2x(1 + x)$

$2x^2 + 2(1 - x)^2 \sin^2 x + 1 = 2x^2 + 2x$

$2(1 - x)^2 \sin^2 x - 2x + 1 = 0.$

Let $h(x) = 2(1 - x)^2 \sin^2 x - 2x + 1$,

clearly $h(1) = -1$

and $h(x) = 2(x^2 - 2x + 1)\sin^2 x - 2x + 1$

$$= x^2\left[2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right).\sin^2 x - \frac{2}{x} + \frac{1}{x^2}\right]$$

$\therefore$ $h(x) \to \infty$ as $x \to \infty$.

$\therefore$ By intermediate value theorem

$h(x) = 0$ has a root which is greater than 1.

Hence Q is true.

24. $$g(x) = \int_1^x \left(\frac{2(t-1)}{(t+1)} - \ell nt\right) f(t)dt$$

$$g'(x) = \left(\frac{2(x-1)}{x+1} - \ell nx\right) f(x)$$

$f(x) > 0 \quad \forall \ x \in R$

Suppose.

$$h(x) = \frac{2(x-1)}{x+1} - \ell nx$$

$$h(x) = 2 - \left(\frac{4}{x+1} + \ell nx\right)$$

$$h'(x) = \frac{4}{(x+1)^2} - \frac{1}{x}$$

$$h'(x) = -\frac{(x-1)^2}{x(x+1)^2}$$

$h'(x) < 0$

So $h(x)$ is decreasing

so $h(x) < h(1)$. $\quad \forall\ x > 1$

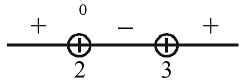
$h(x) < 0 \quad \forall\ x > 1$

So $g'(x) = h(x)\ f(x)$

$g'(x) < 0 \quad \forall\ x > 1$

$g(x)$ is decreasing in $(1, \infty)$.

25. $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$



$\Rightarrow \quad f'(x) = e^{x^2}(x-2)(x-3)$

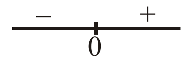
$\therefore \quad f'(2) = f'(3) = 0$

$\Rightarrow \quad f''(c) = 0$ for same $c \in (2,3)$ (by Rolle's theorem)

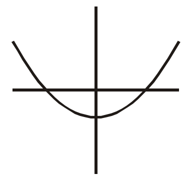
26. $f(x) = x^2 - x\sin x - \cos x$

$f'(x) = 2x - x\cos x - \sin x + \sin x$

$= x(2 - \cos x)$



$\therefore$ graph of $f(x)$ will be



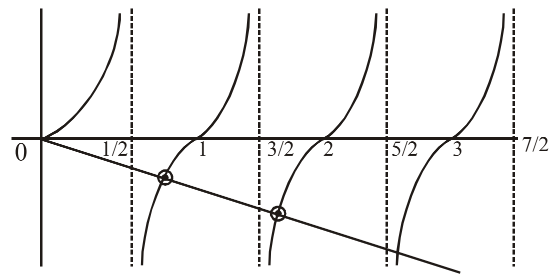
$\therefore \quad f(x)$ is zero for 2 values of x

$\therefore \quad$ (c)

27. $f'(x) = \sin\pi x + \pi x \cos\pi x = 0$

$\tan \pi x = -\pi x$

$y = \tan x\pi$ & $y = -\pi x$



intersection point lies in

$$\left(\frac{1}{2},1\right) \cup \left(\frac{3}{2},2\right) \cup \left(\frac{5}{2},3\right) \ldots$$

option (B) is correct for $\left(n+\frac{1}{2},n\right)$

as well $(n, (n+1))$ because root lies in

$(0,1) \cup (1,2) \cup (2,3)$