DIFFERENTIABILITY

EXERCISE - 01 CHECK YOUR GRASP

2. $f'(0^-) = \lim\limits_{h\to 0} \dfrac{-h\left(\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right)-0}{-h} = -1$

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{h\left(\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right)}{h} = 1$

since $f'(0^-) \neq f'(0^+)$

so f(x) is not differentiable.

5. $f'(0^+) = \lim\limits_{h\to 0} \dfrac{h+h-[h]+h\sin(h-[h])}{h}$

$= \lim\limits_{h\to 0} \dfrac{2h+h\sin h}{h} = 2$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{-h-h-[-h]-h\sin(-h-[-h])}{-h}$

$= \lim\limits_{h\to 0} \dfrac{-2h+1-h\sin(-h+1)}{-h}$

$= \lim\limits_{h\to 0} -2+\dfrac{1}{h}-\sin(1-h)$

$\Rightarrow$ LHD does not exist

hence function is non differentiable and discontinuous at x = 0. Similarly for x = 2.

7. $f(x) = \begin{cases} \dfrac{1}{2x-5} & ; \ x \neq 1 \\ \dfrac{-1}{3} & ; \ x = 1 \end{cases}$

$f'(x) = \dfrac{-2}{(2x-5)^2} \quad \Rightarrow \quad f'(1) = \dfrac{-2}{9}$

10. $f'(x) = \begin{cases} 8x-2 & ; \ \dfrac{-1}{2} \le x < 0 \\ 2ax-b & ; \ 0 \le x < \dfrac{1}{2} \end{cases}$

Now at x = 0

$8(0) - 2 = 2(a)(0) - b \ \Rightarrow \ b = 2$ and $a \in R$

Also f(x) is continuous in $\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$

13. Since [x] is not continuous at integers so x[x] is also not continuous at finite number of points in [– 1, 3] & hence not continuous.

14. $f(x) = \begin{cases} \dfrac{2(\sin x-\sin^3 x)+(\sin x-\sin^3 x)}{2(\sin x-\sin^3 x)-(\sin x-\sin^3 x)} & ; \ x \neq \dfrac{\pi}{2} \end{cases}$

($\because$ $\sin x > \sin^3 x$ in $(0, \pi)$)

$= 3 \ ; \quad x = \dfrac{\pi}{2}$

Now $f(x) = 3 \ ; \quad x \neq \dfrac{\pi}{2}$

$= 3 \ ; \quad x = \dfrac{\pi}{2}$

Hence f(x) is continuous & differentiable at $x = \dfrac{\pi}{2}$

16. $\lim\limits_{h\to 0} |f(x+h) - f(x)| \le (x+h-x)^2$

$\Rightarrow \lim\limits_{h\to 0} |f(x+h) - f(x)| \le |h|^2$

$\Rightarrow \lim\limits_{h\to 0} \left|\dfrac{f(x+h)-f(x)}{h}\right| \le 0 \Rightarrow f'(x) = 0$

$\Rightarrow$ f(x) is constant function $\quad \Rightarrow f(1) = 0$

19. $f(x) = \begin{cases} (x+1)(2x-1), & x < -1 \\ (x+1)(1-2x), & -1 \le x \le 0 \\ x+1, & 0 \le x < 1 \\ (x+1)(2x-1), & x \ge 1 \end{cases}$

$f'(x) = \begin{cases} 4x+1, & x < -1 \\ -4x-1, & -1 \le x \le 0 \\ 1, & 0 \le x < 1 \\ 4x+1, & x \ge 1 \end{cases}$

Function is not differentiable at x = –1, 0 and 1.

EXERCISE - 02 BRAIN TEASERS

5. $f(x) = \dfrac{\sin\frac{\pi}{4}}{1} = \dfrac{1}{\sqrt{2}} \quad ; \quad 1 \le x < 2$

$= \dfrac{\sin\frac{\pi}{2}}{2} = \dfrac{1}{2} \quad ; \quad 2 \le x < 3$

Hence f(x) is continuous at $\dfrac{3}{2}$, differentiable at $\dfrac{4}{3}$ & discontinuous at 2.

6. Since $\sin^{-1} x$ and $\cos\dfrac{1}{x}$ are continuous & differentiable in $x \in [-1, 1] - \{0\}$

Now at x = 0

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{(\sin^{-1}(0-h))^2 \cos\left(\dfrac{-1}{h}\right)-0}{-h} = 0$

$$f'(0^+) = \lim_{h\to 0} \frac{(\sin^{-1} h)^2 \cos\left(\frac{1}{h}\right) - 0}{h} = 0$$

Hence LHD = RHD

so f(x) is continuous & differentiable every where in $-1 \le x \le 1$

10. $$f'(0^+) = \lim_{h\to 0} \frac{g(0+h)\cos\left(\frac{1}{0+h}\right) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{g(h)\cos\left(\frac{1}{h}\right)}{h} = 0$$

$$f'(0^-) = \lim_{h\to 0} = \frac{g(0-h)\cos\left(\frac{1}{0-h}\right) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{g(-h)\cos\left(\frac{-1}{h}\right)}{-h} = 0$$

$\therefore$ f'(0) = 0

11. $$H(x) = \begin{cases} \cos x \ ; & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x \ ; & \frac{\pi}{2} < x \le 3 \end{cases}$$

$$H'\left(\frac{\pi^-}{2}\right) = -\sin x = -1$$

$$H'\left(\frac{\pi^+}{2}\right) = -1$$

Hence H(x) is continuous and derivable in [0, 3] & has maximum value 1 in [0, 3]

12. $f(x) = 3(2x + 3)^{2/3} + 2x + 3$

$$f'(x) = \frac{4}{(2x+3)^{1/3}} + 2$$

Now $2x + 3 \ne 0 \quad \Rightarrow x \ne \frac{-3}{2}$

Hence f'(x) is continuous but not differentiable at $x = -3/2$

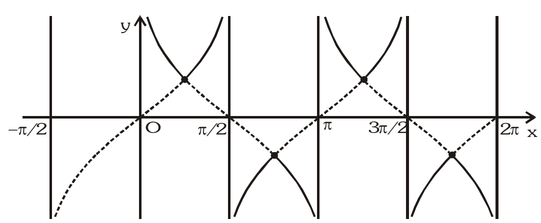
Also f(x) is differentiable & continuous at x = 0

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Match the Column :

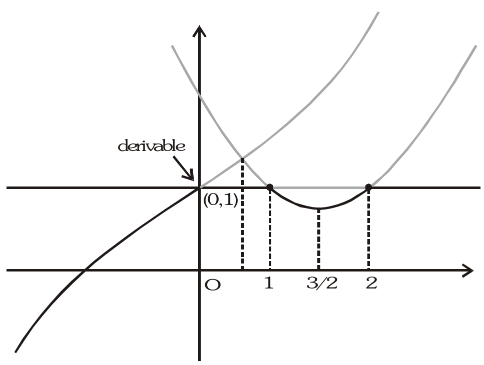
1. (A) $$f(x) = \frac{\tan x + \cot x}{2} - \left|\frac{\tan x - \cot x}{2}\right|$$

$$f(x) = \begin{cases} \cot x \ , & \tan x \ge \cot x \\ \tan x \ , & \tan x < \cot x \end{cases}$$



There are 4 points where the function is continuous but not differentiable in $(0, 2\pi)$

(B)



(C) $f(x) = (x + 4)^{1/3}$

$$f'(x) = \frac{1}{3}(x + 4)^{-2/3}$$

Not derivable at x = –4

(D) $$f(x) = \begin{cases} -\frac{\pi}{2}\ell n\left(\frac{x.2}{\pi}\right) + \frac{\pi}{2}, & 0 < x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

$$f'(x) = \begin{cases} -\frac{\pi}{2x} & 0 < x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

$$f'\left(\frac{\pi^-}{2}\right) = f'\left(\frac{\pi^+}{2}\right) = -1$$

function differentiable at $x = \frac{\pi}{2}$

2. (A) $$f(x) = \begin{cases} 1 - 1 = 0 \ ; & 1 < x \le 2 \\ 0 \ ; & x = 1 \\ 1 - x \ ; & 0 \le x < 1 \\ -\sin \pi x \ ; & -1 \le x < 0 \end{cases}$$

at x = 0, f(x) is not continuous & not differentiable

at x = 1, f(x) is continuous & not differentiable

at x = 2 and – 1, f(x) is continuous & differentiable

(C) $f(x) = \dfrac{x}{x+1}$, not defined at x = – 1

$g(x) = \dfrac{f(x)}{f(x)+2}$

g(x) is not defined at f(x) = – 2

$\dfrac{x}{x+1} = -2 \quad \Rightarrow x = \dfrac{-2}{3}$

Also x = 0 is not in the domain of f(x)

So, at 3 points g(x) is not differentiable.

Comprehension # 2 :

$f(-x) = \dfrac{1}{f(x)}$

But x = 0 $\Rightarrow$ $f^2(0) = 1$ $\Rightarrow$ f(0) = 1

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)f(h)-f(x)}{h}$

$f'(x) = f(x) \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$

$\Rightarrow \dfrac{f'(x)}{f(x)} = -1 \quad \Rightarrow \int \dfrac{f'(x)}{f(x)} dx = -x + c$

$\Rightarrow \ell n\, f(x) = -x + c$

$f(x) = \lambda e^{-x}$

at x = 0, $\lambda = 1$

$\therefore f(x) = e^{-x}$

1. Range of f(x) is R^+
2. Range of f(|x|) is (0, 1]
3. f(x) is decreasing function
4. $f'(x) = -e^{-x} = -f(x)$

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

6. $f(x) = \begin{cases} ax^2 - b & ; \ -1 < x < 1 \\ \dfrac{-1}{x} & ; \ x \ge 1 \\ \dfrac{1}{x} & ; \ x \le -1 \end{cases}$

Now f(x) is differentiable at x = 1

$\Rightarrow 2ax = \dfrac{1}{x^2}$ at x = 1

$\Rightarrow a = \dfrac{1}{2}$

Also f(x) is continuous at x = 1

$\Rightarrow a(1)^2 - b = -1 \Rightarrow b = \dfrac{3}{2}$

9. (a) $f(0^+) = \lim\limits_{h\to 0} (0+h)^m \sin\left(\dfrac{1}{h}\right) = h^m \sin\left(\dfrac{1}{h}\right)$

$\Rightarrow$ m > 0 for continuous

so f(x) is discontinuous if $m \in (-\infty, 0]$

(b) $f'(0^+) = \lim\limits_{h\to 0} \dfrac{(0+h)^m \sin\left(\dfrac{1}{h}\right) - 0}{h}$

$= \lim\limits_{h\to 0} h^{m-1} \sin\left(\dfrac{1}{h}\right)$

$\Rightarrow$ m – 1 > 0 for derivable

$\Rightarrow$ $m \in (0, 1]$, f(x) is continuous but not derivable

10. $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} \Rightarrow 1 = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$

$\therefore \lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1 \ ; \ \lim\limits_{x\to 0} \dfrac{f\left(\dfrac{x}{2}\right)}{\dfrac{x}{2}\times 2} = \dfrac{1}{2}$

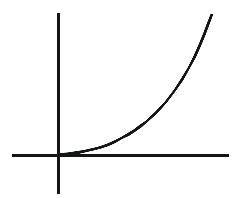
and similarly so on.

On substituting value we get required result.

12. For $t \ge 0$

x = 2t – t = t

$y = t^2 + t^2 = 2t^2$



For t < 0

x = 3t, y = 0

Also when

$0 \le x < 1 \Rightarrow 0 \le t < 1$ [$\therefore$ x = t]

$-1 \le x < 0 \Rightarrow \dfrac{-1}{3} \le t < 0$ [$\therefore$ x = 3t]

f is continuous and differentiable at x = 0

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

3. $f(x + y^n) = f(x) + (f(y))^n$

$f(0 + 0) = f(0) + (f(0))^n \Rightarrow f(0) = 0$

also $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h}$

Let $I = f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+(h^{1/n})^n)-f(0)}{(h^{1/n})^n}$

$= \lim\limits_{h\to 0} \dfrac{f((h^{1/n}))^n}{(h^{1/n})^n} = \lim\limits_{h\to 0} \left(\dfrac{f(h^{1/n})}{h^{1/n}}\right)^n = I^n$

$\Rightarrow I = I^n$ or $I = 0, 1, -1$

since $f'(0) \geq 0$ & f(x) is not identically zero

so $I = 1$ $\therefore f'(0) = 1$ ... (i)

Thus $f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0} \frac{f(x+(h^{1/n})^n)-f(x)}{(h^{1/n})^n}$

$= \lim_{h\to0} \frac{f(x)+(f(h^{1/n}))^n-f(x)}{(h^{1/n})^n}$

$= \lim_{h\to0} \left(\frac{f(h^{1/n})}{h^{1/n}}\right)^n = (f'(0))^n$

$\Rightarrow f'(x) = 1$ (using (i))

Integrating both side

$f(x) = x + c$

$f(x) = x$ $[f(0) = 0]$

$f(10) = 10$

4. $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3}$

$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0} \frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+0}{3}\right)}{h}$

$= \lim_{h\to0} \frac{f(3x)+f(3h)+f(0)-f(3x)-f(0)-f(0)}{3h}$

$= \lim_{h\to0} \frac{f(3h)-f(0)}{3h} = f'(0) \Rightarrow f(x) = xf'(0) + c.$

$\therefore$ f(x) is differentiable for all x in R.

5. $f(1^-) = \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{2[1-h]}{3(1-h)-[1-h]}\right)\right) = \frac{\pi}{2}$

$f(1^+) = \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{2[1+h]}{3(1+h)-[1+h]}\right)\right)$

$= \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{2}{2}\right)\right) = 0$

Hence f(x) is not continuous & not derivable at x = 1

Now at $x = -1$

$f(-1^-) = \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{2[-1-h]}{3(-1-h)-[-1-h]}\right)\right)$

$= \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{-4}{-3+2}\right)\right) = \cos^{-1}1 = 0$

Also $f(-1^+) = \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{2[-1+h]}{3(-1+h)-[-1+h]}\right)\right)$

$= \lim_{h\to0} \cos^{-1}\left(\text{sgn}\left(\frac{-2}{-3+1}\right)\right) = \cos^{-1} 1 = 0$

Hence f(x) is continuous & differentiable at $x = -1$

7. $f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$\geq \lim_{h\to0} \frac{\ell n\left(\frac{x+h}{x}\right)+x+h-x}{h}$

$\geq \lim_{h\to0} \frac{\ell n\left(1+\frac{h}{x}\right)}{h} + 1 \geq \frac{1}{x} + 1$ ....(i)

$f'(x) = \lim_{h\to0} \frac{f(x-h)-f(x)}{-h}$

$\leq \lim_{h\to0} \frac{\ell n\left(\frac{x-h}{x}\right)+x-h-x}{-h}$

$\leq \lim_{h\to0} \frac{\ell n\left(1-\frac{h}{x}\right)}{-h} + 1 \leq \frac{1}{x} + 1$ ....(ii)

from (i) and (ii)

$\Rightarrow f'(x) = \frac{1}{x} + 1$

$\therefore \sum_{n=1}^{100} g\left(\frac{1}{n}\right) = g\left(\frac{1}{1}\right) + g\left(\frac{1}{2}\right) + ... + g\left(\frac{1}{100}\right)$

$= (1 + 2 + 3 + ... 100) + 100 = 5150$

8. (a) $f'(0) = \frac{h^m \sin\left(\frac{1}{h}\right)-0}{h} = h^{m-1} \sin\left(\frac{1}{h}\right)$

$\Rightarrow m - 1 > 0$ for derivable

$f'(x) = mx^{m-1} \sin\left(\frac{1}{x}\right) - x^{m-2} \cos\left(\frac{1}{x}\right)$

f'(x) to be discontinuous at x = 0, $m \in (1, 2]$

(b) Clearly for f(x) to be derivable, & its derivative continuous at x = 0, $m \in (2, \infty)$

1. $f(x + y) = f(x) . f(y)\ \forall\ x, y$

$\therefore\ f(5 + 0) = f(5) . f(0) \quad \{\because f(5) = 2$

$\therefore\ f(0) = 1$

Now $f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$

$= \lim_{h\to 0} \frac{f(5)f(h) - f(5)}{h}$

$= f(5) \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

$= f(5)\ f'(0) = 2 \quad 3 \Rightarrow 6$

4. Apply L Hospital rule

$\lim_{h\to 0} \frac{f'(1+h)}{1} = 5$

$\Rightarrow f'(1) = 5$

5. $|f(x) - f(y)| \le |x - y|^2$

$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \le |x - y|$

$\Rightarrow \lim_{x\to y} \left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{x\to y} |x - y|$

$\Rightarrow f'(x) \le 0 \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is continuous function

$\therefore f(1) = 0 = f(0)$

6. $f(x) = \frac{x}{1+|x|}$ is differentiable

$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ 0, & x = 0 \\ \frac{x}{1+x}, & x > 0 \end{cases}$ L.H.D. $= \lim_{h\to 0} \frac{f(0-h)-f(0)}{h}$

L.H.D.$= \frac{\frac{-h}{1+h} - 0}{-h} = 1$

R.H.D $= \frac{f(0+h)-f(0)}{h} = \frac{\frac{h}{1+h} - 0}{h} = 1$

so differentiable at $(-\infty, \infty)$

7. $gof(x) = \begin{cases} \sin x^2 & ; x \ge 0 \\ -\sin x^2 & ; x < 0 \end{cases}$

$gof(x)$ is continuous (LHL = RHL = 0) = $f(0)$

$gof'(x) = \begin{cases} 2x\cos x^2 & . x > 0 \\ -2x\cos x^2 & ; x < 0 \end{cases}$

LHD = 0 RHD = 0

$gof(x)$ is differentiable

Now $gof''(x) = \begin{cases} 2[\cos x^2 - x\sin x^2 . 2x] & ; x > 0 \\ -2[\cos x^2 - x\sin x^2 . 2x] & ; x < 0 \end{cases}$

LHD = – 2, RHD = 2

Not differentiable.

8. $\lim_{x\to a} \frac{x^2 f(a) - a^2 f(x)}{x - a} \quad \left[\frac{0}{0}\text{ form}\right]$

Use L'Hospital rule

$= \lim_{x\to a} \frac{2x f(a) - a^2 f'(x)}{1}$

$= 2af(a) - a^2 f'(a)$

9. $f(x) = |x - 2| + |x - 5|$; $x \in R$

$f(x)$ is continuous in [2, 5] and differentiable is (2, 5) and $f(2) = f(5) = 3$.

$\therefore$ By Rolle's theorem $f'(x) = 0$ for at least one $x \in (2, 5)$.

$f'(x) = \frac{|x-2|}{x-2} + \frac{|x-5|}{x-5}$

$f'(4) = 0$ but $f'(x) = 0\ \forall\ x \in (2, 5)$

2. Let us first prove that

(I) g is continuous at α and

$f(x) - f(\alpha) = g(x)(x-\alpha), \forall x \in R$

$\Rightarrow$ f(x) is differentiable at α.

Since g is continuous at $x = \alpha$

and $g(x) = \dfrac{f(x)-f(\alpha)}{x-\alpha}$

We should have, $\lim\limits_{x\to\alpha} g(x) = g(\alpha)$

$\Rightarrow \lim\limits_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = g(\alpha) \Rightarrow f'(x) = g(\alpha)$

$\Rightarrow$ $f'(\alpha)$ exists and is equal to $g(\alpha)$.

Conversely now we prove.

(II) f(x) is differentiable at $x = \alpha$

$\Rightarrow$ g is continuous at

$x = \alpha$ and $f(x) - f(\alpha) = g(x)(x-\alpha)\ \forall x \in R$.

$\because$ f(x) is differentiable at $x = \alpha$

$\therefore \lim\limits_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha)$

exists and is finite.

Let us define, $g(x) = \begin{cases} \dfrac{f(x)-f(\alpha)}{x-\alpha}, & x \neq \alpha \\ f'(\alpha), & x = \alpha \end{cases}$

Then, $f(x) - f(\alpha) = (x-\alpha)g(x), \forall x \neq \alpha$

Now for continuity of g(x) at $x = \alpha$

$\lim\limits_{x\to\alpha} g(x) = \lim\limits_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha) = g(\alpha)$

$\therefore$ g is continuous at $x = \alpha$.

4. Given that $f: R \to R$ such that

$f(1) = 3$ and $f'(1) = 6$

Then $\lim\limits_{x\to 0}\left[\dfrac{f(1+x)}{f(1)}\right]^{1/x} = e^{\lim\limits_{x\to 0}\frac{1}{x}[\log f(1+x) - \log f(1)]}$

$= e^{\lim\limits_{x\to 0}\frac{\frac{1}{f(1+x)}f'(1+x)}{1}}$ [Using L' Hospital rule]

$= e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$

5. Given that

$f(x) = \begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x \geq 0 \end{cases} = \begin{cases} x+a \text{ if } x<0 \\ 1-x \text{ if } 0 \leq x < 1 \\ x-1 \text{ if } x \geq 1 \end{cases}$

and $g(x) = \begin{cases} (x+1) & \text{if } x<0 \\ (x-1)^2+b & \text{if } x \geq 0 \end{cases}$

where $a, b \geq 0$

Then $(gof)(x) = g[f(x)]$

$= \begin{cases} f(x)+1 & \text{if } f(x)<0 \\ [f(x)-1]^2+b & \text{if } (f)(x) \geq 0 \end{cases}$

(Using definition of g(x))

Now, $f(x) < 0$ when $x + a < 0$ i.e. $x < -a$

$f(x) = 0$ when $x = -a$ or $x = 1$

$f(x) > 0$ when $-a < x < 1$ or $x > 1$

$g(f(x)) = \begin{cases} f(x)+1 & \text{if } x<-a \\ [f(x)-1]^2+b & \text{if } x=-a \text{ or } x=1 \\ [f(x)-1]^2+b & \text{if } -a<x<0 \\ [f(x)-1]^2+b & \text{if } 0 \leq x < 1 \\ [f(x)-1]^2+b & \text{if } x>1 \end{cases}$

[Keeping in mind that x = 0 and 1 are also the breaking pt's because of definition of f(x)]

$\therefore g[f(x)] = \begin{cases} x+a+1 \text{ if } x<-a \\ (x+a-1)^2+b \text{ if } -a \leq x < 0 \\ ((1-x)-1)^2+b \text{ if } 0 \leq x \leq 1 \\ (x-1-1)^2+b \quad x>1 \end{cases}$

(Substituting the value of f(x) under different conditions)

$\therefore g[f(x)] = \begin{cases} x+a+1 \text{ if } x<-a \\ (x+a-1)^2+b \text{ if } -a \leq x < 0 = F(x)(\text{say}) \\ x^2+b \text{ if } 0 \leq x \leq 1 \\ (x-2)^2+b \text{ if } x>1 \end{cases}$

Now given that $gof(x) \equiv F(x)$ is continuous for all real numbers, therefore it will be continuous at $-a$.

$\Rightarrow$ L.H.L. = R.H.L. = $f(-a)$

$\Rightarrow \lim\limits_{h\to 0} F(-a-h) = \lim\limits_{h\to 0} F(-a+h) = F(-a)$

Now, $\lim\limits_{h\to 0} F(-a-h)$

$= \lim\limits_{h\to 0} a - h + a + 1 = 1$

$\lim\limits_{h\to 0} F(-a+h)$

$= \lim\limits_{h\to 0} (-a+h+a-1)^2 + b = 1 + b$

$F(-a) = 1 + b$

Thus we should have $1 = 1 + b \Rightarrow b = 0$

Again for continuity at $x = 0$

L.H.L. = f(0)

$\Rightarrow \lim\limits_{h\to 0} f(0-h) = f(0)$

$\Rightarrow \lim\limits_{h\to 0} (-h+a-1)^2 + b = b$

$\Rightarrow$ $(a-1)^2 = 0 \Rightarrow a = 1$

For a = 1 and b = 0, gof becomes

$$\text{gof}(x) = \begin{cases} x+2, & x<-1 \\ x^2, & -1 \le x \le 1 \\ (x-2)^2 & x>1 \end{cases}$$

Now to check differentiability of gof(x) at x = 0

We see gof (x) = x^2 = F(x)

$\Rightarrow$ F'(x) = 2x which exists clearly at x = 0

Hence gof is differentiable at x = 0.

6. Given that f : [– 2a, 2a] $\to$ R

f is an odd function.

Lf' at x = a is 0.

$$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{h} = 0 \quad ....(1)$$

To find Lf' at x = – a which is given by

$$\lim_{h\to 0} \frac{f(-a-h)-f(-a)}{-h} = \lim_{h\to 0} \frac{-f(a+h)+f(a)}{-h}$$

[$\because$ f(– x) = – f(x)]

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Again for x $\in$ [a, 2a]

f(x) = f(2a – x)

$\therefore$ f(a + h) = f(2a – a – h) = f(a – h)

substituting this values in last expression we get

$$\text{Lf}'(-a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{h} = 0$$

[Using eq^n (1)]

Hence Lf' (– a) = 0

9. p = –1 Now

$$\lim_{x\to 1^+} \frac{(x-1)^n}{\log\cos^m(x-1)} = \lim_{x\to 1^+} \frac{(x-1)^n}{\log\left[\cos^m(x-1)-1+1\right]}$$

$$= \lim_{x\to 1^+} \frac{(x-1)^n}{\cos^m(x-1)-1}$$

$$= \lim_{x\to 1^+} \frac{n(x-1)^{n-1}}{m\cos^{m-1}(x-1)\sin(x-1)}$$

$$= \frac{-n}{m} \lim_{x\to 1^+} \frac{(x-1)}{\sin(x-1)} \cdot \frac{1}{\cos^{m-1}(x-1)} \times (x-1)^{n-2}$$

$$= \frac{-n}{m} \lim_{x\to 1^+} (x-1)^{n-2} = -1 \text{ (Given)}$$

$\Rightarrow$ n = 2 and m = 2

10. $f(x+y) = f(x) + f(y)$

$f(0) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$f'(x) = f'(0) = k$ (k is constant)

$\Rightarrow$ $f(x) = kx$, hence $f(x)$ is continuous and $f'(x)$ is constant $\forall$ x $\in$ R

11. $f\left(-\frac{\pi}{2}^-\right) = 0$, $f\left(-\frac{\pi}{2}^+\right) = 0$

$$f'(x) = \begin{cases} -1 & x \le \frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x \le 0 \\ 1 & 0 < x \le 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

$f'(0^-) = 0$, $f'(0^+) = 1$ $\therefore$ not differentiable at x = 0

$f'(1^-) = 1$, $f'(1^+) = 1$ $\therefore$ differentiable at x = 1

as $-\frac{3}{2} \in \left(-\frac{\pi}{2}, 0\right)$

$f'(x)$ = sinx which is differentiable at $x = -\frac{3}{2}$

12. At x = 0

$$\text{R.H.D} = \lim_{h\to 0} \frac{(0+h)-(0)}{h} = \lim_{h\to 0} \frac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h}$$

$$= \lim_{h\to 0} h\left|\cos\frac{\pi}{h}\right| = 0 \times \cos(\infty) = 0 \quad \text{finite} = 0$$

$$\text{LHD} : = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{h^2\cos\left(\frac{\pi}{-h}\right) - 0}{-h} = \lim_{h\to 0} -h\cos\left(\frac{\pi}{h}\right)$$

= 0

$\because$ LHD = RHD at x = 0

$\Rightarrow$ $f(x)$ is differentiable at x = 0

At x = 2

$$\text{RHD} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 . \cos\left(\frac{\pi}{2+h}\right) - 0}{h}$$

$$= 4 \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= -4 \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2+h}\right) . \left(-\frac{\pi}{(2+h)^2}\right)}{1} = \pi$$

$$\text{LHD} : \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \left(-\cos\left(\frac{\pi}{2-h}\right)\right)}{-h}$$

$$= \lim_{h\to 0} \frac{4\left(\sin\frac{\pi}{(2-h)}\right)\left(\frac{\pi}{(2-h)^2}\right)}{-1} = -\pi.$$

LHD $\neq$ RHD at x = 2

$\therefore$ Not differentiable at x = 2.