DIFFERENTIAL EQUATION

EXERCISE - 01 CHECK YOUR GRASP

7. $y = e^{mx}$ then $D(y) = me^{mx}$, $D^2(y) = m^2e^{mx}$

$D^3(y) = m^3\ e^{mx}$

then given

$D^3\ y - 3D^2y - 4Dy + 12y = 0$

$\Rightarrow m^3\ e^{mx} - 3m^2e^{mx} - 4me^{mx} + 12\ e^{mx} = 0$

$\Rightarrow m^3 - 3m^2 - 4m + 12 = 0$ ($\because e^{mx} \neq 0$)

$\Rightarrow (m - 2)\ (m - 3)\ (m + 2) = 0$

$\Rightarrow m = 2, 3, -2$

Hence number of values of $m \in N$ will be 2.

12. $\frac{dy}{dx} = 100 - y$

$\Rightarrow \int \frac{dy}{100-y} = \int dx \Rightarrow -\ \ell n\ (100 - y) = x + c$

since $y(0) = 50 \Rightarrow -\ \ell n\ 50 = c$

$\therefore -\ \ell n(100 - y) = x - \ell n\ 50$

$\Rightarrow \ell n\left(\frac{50}{100-y}\right) = x \Rightarrow \frac{50}{100-y} = e^x$

13. $\int_0^x ty(t)dt = x^2y(x)$

Differentiating, we get

$xy = 2xy + x^2 \frac{dy}{dx} \Rightarrow x^2 \frac{dy}{dx} + xy = 0$

$x \frac{dy}{dx} + y = 0 \Rightarrow xdy + ydx = 0$

$d(xy) = 0 \Rightarrow xy = c$

$\therefore$ since it passes through (2,3)

$\therefore c = 6$

Hence $xy = 6$.

14. $\frac{dy}{dx} = \frac{y}{x} - \cos^2 \frac{y}{x}$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = v - \cos^2 v$

$\Rightarrow \int \frac{dv}{\cos^2 v} = \int -\frac{dx}{x}$

$\Rightarrow \tan v = -\ \ell nx + c \Rightarrow \tan\frac{y}{x} = -\ell nx + c$

As it passes through the point $(1, \frac{\pi}{4})$

so $c = 1$

$\tan\frac{y}{x} = -\ \ell nx + 1 \Rightarrow \tan\frac{y}{x} = \ell n\ \frac{e}{x}$

$\therefore y = x\ \tan^{-1}(\ell n\ \frac{e}{x})$

16. $(1 + y^2)\frac{dx}{dy} + x = e^{\tan^{-1} y}$

$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1} y}}{1+y^2}$

I.F. $= e^{\tan^{-1} y}$

$xe^{\tan^{-1} y} = \int \frac{(e^{\tan^{-1} y})^2}{1+y^2}\ dy$

$\Rightarrow xe^{\tan^{-1} y} = \frac{1}{2}\ e^{2\tan^{-1} y} + c$

19. $y\ dx + x\ dy = -\ x^2y\ dy$

$\Rightarrow \frac{ydx + xdy}{x^2y^2} = -\frac{1}{y}dy$

$\Rightarrow \int \frac{d(xy)}{x^2y^2} = -\int \frac{1}{y}dy$

$-\frac{1}{xy} = -\ \ell ny + c \Rightarrow -\frac{1}{xy} + \ell ny = c$

20. $y^5x + y - x\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = y^5$

$\Rightarrow \frac{1}{y^5}\frac{dy}{dx} - \frac{1}{x}\frac{1}{y^4} = 1$

Let $\frac{1}{y^4} = t \Rightarrow \frac{-4}{y^5}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore -\frac{1}{4}\frac{dt}{dx} - \frac{t}{x} = 1$

$\Rightarrow \frac{dt}{dx} + \frac{4t}{x} = -\ 4$

I.F. $= e^{\int \frac{4}{x}dx} = e^{4\ \ell nx} = x^4$

so solution is $t.x^4 = -\int 4.x^4dx + c$

$\Rightarrow \frac{1}{4}\left(\frac{x}{y}\right)^4 + \frac{x^5}{5} = c$

21. $\frac{x}{x^2+y^2}dy = \left(\frac{y}{x^2+y^2} - 1\right)dx$

$\Rightarrow \frac{xdy - ydx}{x^2+y^2} = -dx$

$\Rightarrow \tan^{-1}\frac{y}{x} = -x + c \Rightarrow y = x\ \tan\ (c - x)$

EXERCISE - 02 BRAIN TEASERS

3. Given $\frac{dy}{dt} = -k\sqrt{y}$ $\Rightarrow \int \frac{1}{\sqrt{y}}dy = -k\int dt$

$2\sqrt{y} = -kt + c$

Now at $t = 0$, $y = 4$ so $c = 4$.

$\therefore 2\sqrt{y} = \frac{-t}{15} + 4$ (as $k = \frac{1}{15}$)

When $y = 0$, $t = 60$ min.

5. $y = \frac{x}{\ell n|cx|}$

$$\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

$$\Rightarrow \frac{\ell n|cx|-1}{(\ell n|cx|)^2} = \frac{1}{\ell n|cx|} + \phi\left(\frac{x}{y}\right)$$

$$\Rightarrow \phi\left(\frac{x}{y}\right) = \frac{-1}{(\ell n|cx|)^2} = \frac{-y^2}{x^2}$$

6. $\int_a^x y(t)dt = x^2 + y(x)$

$\Rightarrow xy = 2x + \frac{dy}{dx}$ $\Rightarrow x(y-2) = \frac{dy}{dx}$

$\Rightarrow \int x dx = \int \frac{dy}{y-2}$ $\Rightarrow \frac{x^2}{2} = \ell n\ |y-2| + \ell nc$

$\Rightarrow e^{\frac{x^2}{2}} = c(y-2)$

at $x = a$ $\quad y = -a^2$

$\therefore e^{\frac{a^2}{2}} = c(-a^2-2) \Rightarrow c = -\frac{e^{\frac{a^2}{2}}}{(a^2+2)}$

$\therefore e^{\frac{x^2}{2}} = -\frac{e^{\frac{a^2}{2}}}{(a^2+2)}(y-2)$

$\Rightarrow -y + 2 = (a^2+2)\,e^{\frac{x^2-a^2}{2}}$

$\Rightarrow y = 2 - (2+a^2)\,e^{\frac{x^2-a^2}{2}}$

7. $\int_0^1 f(tx)dt = nf(x)$

Let $tx = u$ $\Rightarrow dt = \frac{du}{x}$

$\therefore \frac{1}{x}\int_0^x f(u)du = nf(x) \Rightarrow \int_0^x f(u)du = nxf(x)$

$f(x) = n[f(x) + xf'(x)] \Rightarrow f(x)\left(\frac{1-n}{n}\right) = xf'(x)$

$\int \frac{dx}{x} = \frac{n}{1-n}\int \frac{dy}{y} \Rightarrow \frac{1-n}{n}\ell nx = \ell ny + \ell nc$

$x^{\frac{1-n}{n}} = cy \Rightarrow y = c'\,x^{\frac{1-n}{n}}$

12. $y = mx + c$ $\Rightarrow \frac{dy}{dx} = m$

It satisfies $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

$m + xm^2 - mx - c = 0$

$x(m^2 - m) + (m - c) = 0$

This is an identity so

$m = 0$ or $m = 1$ & $c = m$

So two such straight line are possible.

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

True/False :

2. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$

$y = e^{-t}$ then $\frac{dy}{dt} = -e^{-t}$ and $\frac{d^2y}{dt^2} = e^{-t}$

$\therefore e^{-t} - 2e^{-t} + e^{-t} = 0$

Hence e^{-t} is a solution of the above equation.

Now $y = te^{-t}$, then

$\frac{dy}{dt} = e^{-t} - te^{-t}$ and $\frac{d^2y}{dt^2} = -2e^{-t} + te^{-t}$

On putting this in equation, te^{-t} also satisfies the equation.

5. $y = ax^2 + bx + c$ $\quad \frac{dy}{dx} = 2ax + b$

$\frac{d^2y}{dx^2} = 2a$ $\Rightarrow \frac{d^3y}{dx^3} = 0$

Hence order 3, degree 1.

Assertion & Reason :

2. $y = A + \ell n\ Bx$ $\quad \therefore \frac{dy}{dx} = \frac{1}{x}$

Hence order is 1.

So statement-I is false & statement-II is true.

Comprehension : # 1

For reservoir A

$\frac{dV_A}{dt} \propto -V_A$ $\Rightarrow \frac{dV_A}{dt} = -k_1 V_A$

$\Rightarrow \int_{V_{A_0}}^{V_A} \frac{dV_A}{V_A} = -k_1\int_0^t dt$ $\Rightarrow \log\frac{V_A}{V_{A_0}} = -k_1 t$

$\Rightarrow V_A = V_{A_0}\,e^{-k_1 t}$

Similarly $V_B = V_{B_0}e^{-k_2 t}$

so $\frac{V_A}{V_B} = \frac{V_{A_0}}{V_{B_0}}\,e^{-(k_1-k_2)t}$

At $t = 0$, $V_{A_0} = 2V_{B_0}$

and at $t = 1$, $V_A = 1.5\ V_B$

so $\frac{3}{2} = 2e^{-(k_1 - k_2)}$ $\quad\therefore\quad e^{-(k_1 - k_2)} = \frac{3}{4}$

1. At $t = \frac{1}{2}$, $V_A = kV_B$

so $k = 2\left(\frac{3}{4}\right)^{1/2} \Rightarrow k = \sqrt{3}$

2. Let at $t = t_0$ both the reservoirs have same quantity of water, then

$V_A = V_B \quad\Rightarrow\quad 2e^{-(k_1 - k_2)t_0} = 1$

$\Rightarrow \left(\frac{3}{4}\right)^{t_0} = \frac{1}{2} \quad\therefore\ t_0 = \log_{3/4}\left(\frac{1}{2}\right)$

$\Rightarrow\quad t_0 = \log_{4/3} 2$

and also $t_0 = \dfrac{1}{\log_2 \frac{4}{3}} = \dfrac{1}{2 - \log_2 3}$

3. Now $\frac{V_A}{V_B} = 2\ e^{-(k_1 - k_2)t} \Rightarrow f(t) = 2\ e^{-(k_1 - k_2)t}$

$f'(t) = -2(k_1 - k_2)\ e^{-(k_1 - k_2)t} = 2\ln\frac{3}{4}\ e^{-(k_1 - k_2)t}$

$\Rightarrow$ f(t) is decreasing.

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

3. $x^2 + y^2 + 2gx + 2fy + c = 0$

Differentiation, we get,

$\Rightarrow\quad 2x + 2yy' + 2g + 2fy' = 0$

Again differentiating,

$\Rightarrow\quad 1 + (y')^2 + yy'' + fy'' = 0 \qquad$ ....(i)

Again differentiating,

$\Rightarrow\quad 2(y')y'' + y'y'' + yy''' + fy''' = 0$

$\Rightarrow\quad 3y'y'' + y'''\left[y - \frac{1}{y''} - \frac{(y')^2}{y''} - y\right] = 0$ (from (i))

$\Rightarrow\quad 3y'(y'')^2 - y'''\left[1 + (y')^2\right] = 0$

5. $y = c_1\ e^{3x} + c_2e^{2x} + c_3e^x \qquad$ ... (1)

$\frac{dy}{dx} = 3c_1\ e^{3x} + 2c_2\ e^{2x} + c_3\ e^x \qquad$ ... (2)

$\frac{d^2y}{dx^2} = 9c_1\ e^{3x} + 4c_2\ e^{2x} + c_3\ e^x \qquad$ ... (3)

$\frac{d^3y}{dx^3} = 27c_1\ e^{3x} + 8c_2\ e^{2x} + c_3\ e^x \qquad$ ... (4)

Apply (4) – 6 (3) + 11 (2) – 6 (1)

$\Rightarrow\quad \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

11. $\frac{xdx - ydy}{xdy - ydx} = \sqrt{\frac{1 + x^2 - y^2}{x^2 - y^2}}$

Put $x = r\sec\theta$, $y = r\tan\theta$

$\therefore\ x^2 - y^2 = r^2 \qquad$ ... (1)

and $\sin\theta = \frac{y}{x} \qquad$ ... (2)

On differentiating (1), $xdx - ydy = rdr$... (3)

On differentiating (2),

$xdy - ydx = x^2\cos\theta\ d\theta = r^2\sec\theta\ d\theta$ ... (4)

Now $\frac{rdr}{r^2\sec\theta d\theta} = \frac{\sqrt{1 + r^2}}{r}$

$\Rightarrow \int\frac{dr}{\sqrt{1 + r^2}} = \int\sec\theta\ d\theta$

$\Rightarrow \ell n(r + \sqrt{1 + r^2}) = \ell n(\sec\theta + \tan\theta) + \ell nc$

$\Rightarrow r + \sqrt{1 + r^2} = c(\sec\theta + \tan\theta)$

$\sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = c\left(\frac{x + y}{\sqrt{x^2 - y^2}}\right)$

12. $\frac{dy}{dx} = \sin\frac{x - y}{2} - \sin\frac{x + y}{2}$

$\Rightarrow\quad \frac{dy}{dx} = -2\cos\frac{x}{2}\sin\frac{y}{2}$

$\Rightarrow\quad \operatorname{cosec}\frac{y}{2}dy = -2\cos\frac{x}{2}dx$

$\Rightarrow\quad 2\left(\ell n\left|\tan\frac{y}{4}\right|\right) = 2\left(-2\sin\frac{x}{2}\right) + c$

$\Rightarrow\quad \ell n\left|\tan\frac{y}{4}\right| = c_1 - 2\sin\frac{x}{2}$

16. $\frac{dm}{dt} = -\lambda m$

$\frac{1}{m}dm = -\lambda dt$

$\ell n\ m = -\lambda t + c$

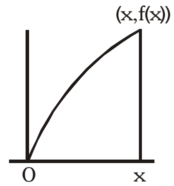
$m = ke^{-\lambda t} \qquad$ (at $t = 0$, $m = m_0$)

$\Rightarrow\quad k = m_0$

$m = m_0e^{-\lambda t} \qquad$ (at $t = t_0$, $m = m_0 - \frac{\alpha m_0}{100}$)

$\Rightarrow\quad \lambda = \frac{-1}{t_0}\ell n\left(1 - \frac{\alpha}{100}\right)$

18. $A = \int_0^x f(t)dt = \lambda(f(x))^{n+1} = \lambda y^{n+1}$



$\Rightarrow \quad y = \lambda(n+1)y^n y'$

$\Rightarrow \quad dx = \lambda(n+1)y^{n-1}dy$

$\Rightarrow \quad x = \frac{\lambda(n+1)}{n}y^n + C$

$\Rightarrow \quad C = 0,\ \lambda = \frac{n}{n+1} \qquad (f(0) = 0,\ f(1) = 1)$

$\Rightarrow \quad x = y^n \Rightarrow y = x^{1/n}$

22. Put $x \to X + h \quad y = Y + k$

$\begin{matrix} h+2k=3 \\ 2h+k=3 \end{matrix} \Rightarrow h = 1,\ k=1$

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$

Put $Y = vX$

$\frac{Xdv}{dX} = \frac{1+2v}{2+v} - v$

$\Rightarrow \quad \frac{2+v}{(1+v)(1-v)}dv = \frac{dX}{X}$

$\Rightarrow \quad \int\left(\frac{1}{1-v} + \frac{1}{1-v^2}\right)dv = \ell n\,|X| + c$

$\Rightarrow \quad -\ell n\,|1-v| + \frac{1}{2}\ell n\left|\frac{v+1}{1-v}\right| = \ell n\,|X| + c$

$\Rightarrow \quad -\ell n\,|X-Y| + \ell n\left|\frac{Y+X}{X-Y}\right|^{1/2} = c$

$\Rightarrow \quad \ell n\left|\frac{\sqrt{\frac{X+Y}{X-Y}}}{X-Y}\right| = c \quad \Rightarrow \quad \frac{X+Y}{(X-Y)^3} = k$

$\Rightarrow \quad (x + y - 2) = k(x - y)^3$

29. $y\sqrt{1+\left(\frac{dx}{dy}\right)^2} + y\frac{dx}{dy} = kxy$

$\Rightarrow \sqrt{1+\left(\frac{dx}{dy}\right)^2} = kx - \frac{dx}{dy}$

$\Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = k^2x^2 + \left(\frac{dx}{dy}\right)^2 - 2kx\frac{dx}{dy}$

$\Rightarrow 2kx\frac{dx}{dy} = k^2x^2 - 1 \qquad \Rightarrow \int\frac{2kxdx}{k^2x^2-1} = \int dy$

$\Rightarrow y = \frac{1}{k}\ \ell n\ |c(k^2x^2 - 1)|$

30. $y^3\frac{dy}{dx} + x + y^2 = 0$

Put $y^2 = a - x \quad \Rightarrow \quad 2y.\frac{dy}{dx} = \frac{da}{dx} - 1$

so $\frac{1}{2}\left(\frac{da}{dx} - 1\right)(a - x) + x + a - x = 0$

$\Rightarrow \quad \frac{1}{2}\left(\frac{da}{dx} - 1\right) = \frac{-a}{a-x}$

$\Rightarrow \quad \frac{da}{dx} - 1 = \frac{2a}{x-a} \quad \Rightarrow \frac{da}{dx} = \frac{2a}{x-a} + 1$

$\therefore \quad \frac{da}{dx} = \frac{x+a}{x-a}$

Put $a = vx$

so $v + x\frac{dv}{dx} = \frac{1+v}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{1-v} \qquad \Rightarrow \int\frac{(1-v)dv}{1+v^2} = \int\frac{dx}{x}$

$\Rightarrow \quad \tan^{-1} v - \frac{1}{2}\ \ell n\ |1 + v^2| = \ell nx + c$

$\Rightarrow \quad \tan^{-1}\frac{a}{x} - \frac{1}{2}\ \ell n\left|1+\frac{a^2}{x^2}\right| = \ell nx + c$

$\Rightarrow \quad \tan^{-1}\frac{a}{x} - \frac{1}{2}\ \ell n\ |x^2 + a^2| = c$

where $a = x + y^2$

35. $\left[1+y(1+x^2)\right]dx = -x(1+x^2)dy$

$\frac{dy}{dx} + \frac{y}{x} = -\frac{1}{x(1+x^2)}$

I.F. $\quad e^{\int 1/x} = x$

$\therefore \quad yx = -\int\frac{1}{1+x^2}dx$

$xy = -\tan^{-1}x + c$

36. $y - xy' = b + bx^2y'$

$\Rightarrow \quad y - b = (x + bx^2)\frac{dy}{dx} \Rightarrow \frac{dx}{x(1+bx)} = \frac{dy}{y-b}$

$\Rightarrow \quad \int\left(\frac{1}{x} - \frac{b}{1+bx}\right)dx = \log(y-b) + \log c$

$\Rightarrow \quad \log x - \log(1 + bx) = \log(y - b) + \log c$

$\Rightarrow \quad \frac{x}{c} = (1 + bx)(y - b)$

$\Rightarrow \quad b + (\frac{1}{c} + b^2)x = y(1 + bx)$

$\Rightarrow \quad b + kx = y(1 + bx)$

40. $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^4 \cos x}{x^3}$

$\frac{1}{y^4}\frac{dy}{dx} - \frac{1}{y^3 x} = -\frac{\cos x}{x^3}$

Put $-\frac{1}{y^3} = t \quad \Rightarrow \quad \frac{3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad \frac{dt}{dx} + \frac{3t}{x} = \frac{-3\cos x}{x^3}$

I.F. $\quad e^{\int 3/x} = x^3$

$tx^3 = -\int \frac{3\cos x}{x^3} x^3 \, dx \quad \Rightarrow \quad -\frac{1}{y^3}x^3 = -3\sin x + c$

$\Rightarrow \quad x^3 y^{-3} = 3\sin x - c$

41. $\frac{dy}{dx} - y = \frac{2xy^2}{e^x} \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \frac{2x}{e^x}$

Put $-\frac{1}{y} = t \quad \Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad \frac{dt}{dx} + t = \frac{2x}{e^x}$

I.F. is $e^{\int dx} = e^x$

$te^x = \int \frac{2x}{e^x} e^x \, dx \quad \Rightarrow \quad -\frac{1}{y}e^x = x^2 + c$

$\Rightarrow \quad y^{-1}e^x = -x^2 - c$

43. $x(x^2 + 1)\frac{dy}{dx} + (x^2 - 1)y = x^3 \ell nx$

$$\frac{dy}{dx} + \frac{(x^2-1)y}{x(x^2+1)} = \frac{x^2 \ell n x}{x^2+1}$$

$$\text{I.F.} = e^{\int \frac{x^2-1}{x(x^2+1)}dx} = \frac{x^2+1}{x}$$

$$\left(\frac{x^2+1}{x}\right)y = \int \frac{x^2+1}{x} \quad \frac{x^2}{x^2+1} \ell nx \, dx$$

$$\Rightarrow \quad \left(\frac{x^2+1}{x}\right)y = \int x \; \ell nx \, dx$$

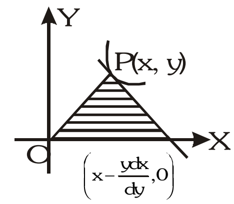
$$\Rightarrow \quad \left(\frac{x^2+1}{x}\right)y = \ell nx . \frac{x^2}{2} - \frac{x^2}{4} + c$$

$\Rightarrow \quad 4(x^2 + 1)y + x^3(1 - 2\ell nx) = cx$

46. Equation of tangent

$Y - y = \frac{dy}{dx}(X - x)$

when $Y = 0$, $X = x - y\frac{dx}{dy}$



$\left|\frac{1}{2}\left(x - y\frac{dx}{dy}\right)y\right| = a^2$

$xy - y^2\frac{dx}{dy} = \pm 2a^2 \quad \Rightarrow \quad \frac{dx}{dy} - \frac{x}{y} = \mp \frac{2a^2}{y^2}$

$\Rightarrow \frac{x}{y} = \mp \int \frac{2a^2}{y^2} \; \frac{1}{y} \, dy \quad \Rightarrow \frac{x}{y} = \pm \frac{a^2}{y^2} + c$

$\Rightarrow x = cy \pm \frac{a^2}{y}$

52. $\left(\frac{dy}{dx} - y\right)\left(\frac{dy}{dx} - x\right) = 0$

$\frac{dy}{dx} = y \qquad \text{or} \qquad \frac{dy}{dx} = x$

$\Rightarrow \quad \ell ny = x + c \quad \text{or} \quad y = \frac{x^2}{2} + c$

$\Rightarrow \quad y = ke^x$

54. $xy = t \quad \Rightarrow \quad x\frac{dy}{dx} + y = \frac{dt}{dx}$

$\Rightarrow \quad x\frac{dy}{dx} = \frac{dt}{dx} - \frac{t}{x}$

$\Rightarrow \quad (1 - t + t^2) = x\left(\frac{dt}{dx} - \frac{t}{x}\right)$

$\Rightarrow \quad \frac{(1 - t + t^2)}{x} + \frac{t}{x} = \frac{dt}{dx} \quad \Rightarrow \quad \frac{1 + t^2}{x} = \frac{dt}{dx}$

$\Rightarrow \quad \frac{dt}{1 + t^2} = \frac{dx}{x} \quad \Rightarrow \quad \tan^{-1}t = \ell n|x| + \ell nc$

$\Rightarrow \quad \tan^{-1}(xy) = \ell n|cx|$

$xy = \tan \ell n|xc|$

57. Equation of tangent $Y - y = \frac{dy}{dx}(X - x)$

$$\text{distance from origin} = \left|\frac{-x\frac{dy}{dx} + y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

Equation of normal $\qquad Y - y = \frac{-1}{\frac{dy}{dx}}(X - x)$

$\Rightarrow Y\frac{dy}{dx} - y\frac{dy}{dx} = -X + x$

distance from origin $= \left|\frac{x + y\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$

Now $\left|-x\frac{dy}{dx}+y\right| = \left|x+y\frac{dy}{dx}\right|$

either $-x\frac{dy}{dx} + y = x + y\frac{dy}{dx}$

or $x\frac{dy}{dx} - y = x + y\frac{dy}{dx}$

$\Rightarrow (x + y)\frac{dy}{dx} = y - x$

or $(x - y)\frac{dy}{dx} = x + y$

$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$ or $\frac{dy}{dx} = \frac{x+y}{x-y}$

Put $y = vx$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$ or $v + x\frac{dv}{dx} = \frac{1+v}{1-v}$

$\Rightarrow \int\frac{v+1}{1+v^2}\,dv = \int\frac{-dx}{x}$ or $\int\frac{v-1}{1+v^2}\,dv = \int\frac{-dx}{x}$

$\Rightarrow \frac{1}{2}\ell n\ |1+v^2| + \tan^{-1} v = -\ell nx + \ell nc$

or $\frac{1}{2}\ell n\ |1+v^2| - \tan^{-1} v = \ell nc - \ell nx$

Hence solution will be

$\frac{1}{2}\ell n\ |1+v^2| + \ell nx = \pm\tan^{-1} v + \ell nc$

$x\sqrt{1+v^2} = ke^{\pm\tan^{-1} v}$

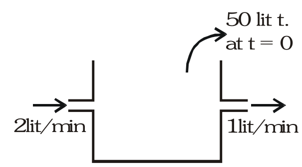
$\Rightarrow \sqrt{x^2+y^2} = ce^{\pm\tan^{-1} y/x}$

58. Input rate = 10 gm/min

After time t volume of tank = 50 + (2 − 1)t

Concentration of salt in time t $= \frac{m}{50+t}$ gm/lit

Output rate $= \frac{m}{50+t}.1$ gm/min.



$\frac{dm}{dt} = \frac{-m}{50+t} + 10$

$m(50 + t) = \int 10(50 + t)\,dt$

$\Rightarrow m(50 + t) = 500\,t + 5t^2 + c$

$\Rightarrow (\because$ at $t = 0,\ m = 0 \Rightarrow c = 0)$

$\Rightarrow m(50 + t) = 5\,(100\,t + t^2)$

$\Rightarrow m = 5t\left(\frac{100+t}{50+t}\right)$ gm

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. **(a)** $\frac{du}{dx} + Pu = Q;\ \frac{dv}{dx} + Pv = Q$

$\Rightarrow \frac{d}{dx}(u - v) = -P(u - v)$

$\frac{d(u-v)}{u-v} = -P\,dx$

$\Rightarrow \ell n\ (u - v) = -\int Pdx$

$\frac{dy}{dx} + Py = Qx$

I.F. $= \frac{1}{u-v}$

$y.\frac{1}{u-v} = \int\frac{Q}{u-v} + c$

$\Rightarrow \frac{u}{u-v} = \int\frac{Q}{u-v} + c'$ [u satisfies it]

$\therefore \frac{y}{u-v} = \frac{u}{u-v} + k$

$\Rightarrow y = u + k\,(u - v)$ ... (1)

(b) If $y = \alpha u + \beta v$ is a particular solution then compare with (1)

$\alpha = k + 1,\ \beta = -k$

$\Rightarrow \alpha + \beta = 1$

(c) If ω is a particular solution then it satisfies (1)

$\Rightarrow \omega = u + k\,(u - v)$

$\frac{u-v}{\omega-u} =$ constant

2. $Y - y = -\frac{1}{y'}(X - x)$

At x axis,

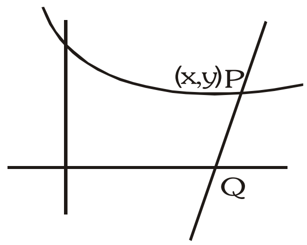
$X = x + y.\,y'$

mid point of PQ = $\left(\frac{2x+yy'}{2}, \frac{y}{2}\right)$

mid point lies on $2y^2 = x$

$\therefore \frac{2y^2}{4} = \frac{2x+yy'}{2} \Rightarrow \frac{ydy}{dx} - y^2 = -2x$

Put $y^2 = t$



$\therefore 2y\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{1}{2}\frac{dt}{dx} - t = -2x$

$\frac{dt}{dx} + (-2)t = -4x$

$\therefore t\, e^{-2x} = -4\int e^{-2x}.x\, dx + c$

$\therefore y^2 e^{-2x} = e^{-2x}(2x+1) + c$

$x = 0,\ y = 0 \Rightarrow c = -1$

$\therefore y^2 = 2x + 1 - e^{2x}$

6. $f(x) > 0 \quad \forall\ x \geq 2$

$\frac{d}{dx}(x\, f(x)) \leq -k\, f(x)$

$x\frac{dy}{dx} + y \leq -ky \qquad [f(x) = y]$

$x\frac{dy}{dx} \leq -y(k+1)$

$\frac{dy}{dx} + \frac{(k+1)}{x}y \leq 0$

I.F. $= e^{\int\frac{k+1}{x}dx} = e^{\ell n\, x^{k+1}} = x^{k+1}$

$x^{k+1}.\frac{dy}{dx} + (k+1).x^k y \leq 0$

$\frac{d}{dx}(y.x^{k+1}) \leq 0$

$\Rightarrow g(x) = y.x^{k+1}$ decreases $\forall\ x \geq 2$

$\therefore g(x) \leq f(2).2^{k+1}$

$f(x).x^{k+1} \leq f(2).2^{k+1}$

$f(x) \leq A.x^{-k-1}$

8. $\int f(x)dx = F(x) + c$

$f(x) = F'(x)$

let $F(x) = y$

$\frac{dy}{dx} + \cos x.y = \frac{\sin 2x}{(1+\sin x)^2}$

I.F. $= e^{\sin x}$

$y.e^{\sin x} = 2\int\frac{e^{\sin x}.\sin x\cos x}{(1+\sin x)^2}dx + c$

$y\, e^{\sin x} = 2\int\frac{e^t(t+1-1)}{(1+t)^2}dt \qquad$ [Put $\sin x = t$]

$= 2\int e^t\left\{\frac{1}{t+1} + \frac{-1}{(1+t)^2}\right\} + c$

$y\, e^{\sin x} = \frac{2e^t}{t+1} + c \Rightarrow y\, e^{\sin x} = \frac{2e^{\sin x}}{\sin x+1} + c$

$\Rightarrow y = \frac{2}{\sin x+1} + c.\, e^{-\sin x}$

$f(x) = \frac{dy}{dx} = \frac{-2\cos x}{(\sin x+1)^2} - c.e^{-\sin x}\cos x$

13. $\frac{dy_1}{dx} + Py_1 = Q,\ \frac{dy_2}{dx} + Py_2 = Q$

Put $y_2 = y_1 z$

$\Rightarrow \frac{dy_2}{dx} = y_1\frac{dz}{dx} + z\frac{dy_1}{dx}$

$\Rightarrow y_1\frac{dz}{dx} + z\frac{dy_1}{dx} = Q - Py_2$

$\Rightarrow y_1\frac{dz}{dx} + z\frac{dy_1}{dx} + Py_1 z = Q$

$\Rightarrow y_1\frac{dz}{dx} + zQ = Q \Rightarrow y_1\frac{dz}{dx} = Q(1-z)$

$\Rightarrow \int\frac{dz}{1-z} = \int\frac{Q}{y_1}dx$

$\Rightarrow \ell n\,|z-1| = -\int\frac{Q}{y_1}dx + \lambda$

$\Rightarrow z = 1 + a\, e^{-\int\frac{Q}{y_1}dx}$

1. $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ Put $y = vx \Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{v^2x^2-x^2}{2x.vx} = \frac{v^2-1}{2v}$

or $\frac{xdv}{dx} = \frac{v^2-1}{2v} - v$

$x\frac{dv}{dx} = \frac{-v^2-1}{2v}$ or $-\int\frac{2v\,dv}{1+v^2} = \int\frac{dx}{x}$

$-\log(1 + v^2) = \log x + c$

$\log x + \log(1 + v^2) = \log c$

$\log x.\left(1+\frac{y^2}{x^2}\right) = \log c$ or $x\left(\frac{x^2+y^2}{x^2}\right) = c$

$\frac{x^2+y^2}{x} = c$ or $x^2 + y^2 = cx$

2. $y = e^{cx}$

$\log y = cx$ ..... (i)

$\frac{1}{y}y' = c \Rightarrow y' = cy$

$c = \frac{y'}{y}$ put in equation (i) $\log y = \frac{y'}{y}.x$

or $y \log y = xy'$

3. Given $\frac{dy}{dx} = \frac{y-1}{x(x-1)}$ or $\int\frac{dy}{y-1} = \int\frac{dx}{x(x+1)}$

$\log(y - 1) = \log\left(\frac{x}{x+1}\right) + \log C$

or $y - 1 = \frac{cx}{x+1}$ ..... (i)

Equation (i) passes through (1, 0)

$-1 = \frac{C}{2} \Rightarrow C = -2$ Put in (i)

$(y - 1) = \frac{-2x}{x+1}$ $(y - 1)(x + 1) + 2x = 0$

4. Equation of given parabola is $y^2 = Ax + B$ where A and B are parameters

$2y\frac{dy}{dx} = A$ $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

This is the equation of given parabola order = 2, degree 1

5. $(1 + y^2) = (e^{\tan^{-1}y} - x)\frac{dy}{dx}$ or $(1+ y^2)\frac{dx}{dy} + x + e^{\tan^{-1}y}$

$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{(1+y^2)}$

Now I.F. $= e^{\int\frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

$\therefore$ solution $x(e^{\tan^{-1}y}) = \int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}\,dy + C$

$xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + C$

or $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + K$

6. Given family of curves is

$x^2 + y^2 - 2ay = 0$ ..... (1)

$2x + 2yy' - 2ay' = 0$ ..... (2)

Now put the value of 2a from (1) to in (2)

$2x + 2yy' - \frac{x^2+y^2}{y}.y' = 0$

$2xy + (y^2 - x^2)y' = 0$ or $(x^2 - y^2)y' = 2xy$

7. $ydx + (x + x^2y)dy = 0$ $ydx + xdy = -x^2y\,dy$

$\int\frac{d(xy)}{(xy)^2} = -\int\frac{dy}{y} \Rightarrow -\frac{1}{xy} = -\log y + c$

$\Rightarrow \frac{-1}{xy} + \log y = c$

8. $y^2 = 2c(x + \sqrt{c})$ ..... (1)

$y^2 = 2cx + 2c\sqrt{c}$

$2y\frac{dy}{dx} = 2c \Rightarrow yy_1 = C$ Put in equation(1)

$\Rightarrow y^2 = 2yy_1(x + \sqrt{yy_1})$

$y^2 = -2yy_1x = 2yy_1\sqrt{yy_1}$ or $(y^2 - 2yy_1x)^2 = 4y^3y_1^3$

Degree = 3 order = 1

9. $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x}+1\right)$ which is homogeneous equ.

Put $y = vx$, $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$v + x\frac{dv}{dx} = \frac{vx}{x}\left(\log\frac{vx}{x}+1\right)$

$\frac{xdv}{dx} = v(\log v + 1) - v = v\log v + v - v$

$\int\frac{dv}{v\log v} = \int\frac{dx}{x} \Rightarrow \log(\log v) = \log x + \log c$

$\Rightarrow \log\frac{y}{x} = cx$

10. Given $Ax^2 + By^2 = 1$ Divide by B

$$\frac{A}{B}x^2 + y^2 = \frac{1}{B} \quad \text{Differentiate w.r.t x}$$

$$2x\frac{A}{B} + 2y\frac{dy}{dx} = 0 \quad \text{..... (i)}$$

Again Differentiate w.r.t. x

$$2\frac{A}{B} + 2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] = 0 \quad \text{..... (ii)}$$

Put $\frac{A}{B} = -\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]$ in equation (i)

$$-2x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] + 2y\frac{dy}{dx} = 0$$

or $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

It have second order and first degree.

11. Let the centre of circle is (h, 0) and radius will be also h

$\therefore$ equation of circle $(x - h)^2 + (y - 0)^2 = h^2$

$\Rightarrow$ $x^2 - 2hx + h^2 + y^2 = h^2$

$\Rightarrow$ $x^2 - 2hx + y^2 = 0$ ..... (i)

Equation (i) passes through origin differentiating it w.r.t. x

$2x-2h+2y\frac{dy}{dx}=0 \Rightarrow h = x+ y\frac{dy}{dx}$ put in equation (i)

$$x^2 - 2x\left(x+y\frac{dy}{dx}\right) + y^2 = 0$$

$$\Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

12. $\frac{dy}{dx} = 1 + \frac{y}{x}$ put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = 1 + \frac{vx}{x} \Rightarrow x\frac{dv}{dx} = 1$$

$\int dv = \int\frac{dx}{x} \Rightarrow v = \log x + c$ or $\frac{y}{x} = \log x+c$ ... (i)

Given $y(1) = 1 \Rightarrow 1 = \log 1 + c \Rightarrow c = 1$ put (i)

$y = x\ell nx + x$

13. Equation of circle $(x - h)^2 + (y - 2)^2 = 25$ .....(i)

Differentiate w.r.t. x

$$2(x - h) + 2(y - 2)\frac{dy}{dx} = 0$$

$(x - h) = -(y - 2)\frac{dy}{dx}$ put in (i)

$$(y - 2)^2\left(\frac{dy}{dx}\right)^2 + (y - 2)^2 = 25$$

or $(y - 2)^2 (y')^2 + (y - 2)^2 = 25$

14. $y = c_1e^{c_2X}$ ....(1)

$y' = c_1c_2e^{c_2x}$ ....(2)

$y'' = c_1c_2^2e^{c_2x}$

$y'' = c_2y'$ ....(3)

Now $\frac{(2)}{(1)}$

$\frac{y'}{y} = c_2$

$\Rightarrow$ Put in (3)

$y'' = \frac{y'}{y}.y' \Rightarrow y''\, y = (y')^2$

15. $\cos x\, dy = y(\sin x - y)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin x - y^2}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = y \tan x - y^2 \sec x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{1}{y}\tan x - \sec x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \quad \text{... (1)}$$

Put $\frac{1}{y} = t$ in equation (1)

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx} \quad \text{... (2)}$$

From equation (1) & (2), we get,

$$\Rightarrow \frac{dt}{dx} + t \,.\, \tan x = \sec x$$

$\therefore$ I.F. $= e^{\int \tan x\, dx}$

$= e^{\log|\sec x|} = \sec x$

$\therefore$ solution of differential equation is :

t. $\sec x = \int \sec x \,.\, \sec x \,.\, dx + c$

$\frac{1}{y}\sec x = \tan x + c$

$\sec x = y(\tan x + c)$

16. $\frac{dy}{dx} = y + 3 > 0$ $y(0) = 2$, $y(\log 2) = ?$

$$\int\frac{dy}{y+3} = \int dx$$

$\log|y + 3| = x + c$

y(0) = 2

$\log|2+3| = 0 + c \Rightarrow c = \log 5.$

$y.(\log 2) = ?$

$\log|y+3| = \log 2 + \log 5$

$\log|y+3| = \log 10$

$y + 3 = 10$

$y = 7$

17. $\frac{dV}{dt} = -k(T-t)$

$\int dV = \int -K(T-t)dt$

$V = -K\left[Tt - \frac{t^2}{2}\right] + C$

At $t = 0$ $V = I \Rightarrow C = I$

$V = -Kt\left(T - \frac{t}{2}\right) + I$

$V(T) = -KT\left(T - \frac{T}{2}\right) + I = \frac{-KT^2}{2} + I$

18. Equation of tangent at (x_1, y_1) is

$y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$

x– intercept $= x_1 - y_1\frac{dx_1}{dy_1}$

According to question

$x_1 = \frac{x_1 - y_1\frac{dx_1}{dy_1}}{2}$

$\Rightarrow x_1 = -y_1\frac{dx_1}{dy_1}$

$\int\frac{dy}{y} = \int -\frac{dx}{x}$

$\Rightarrow \ell ny = -\ell nx + \ell nc$

$\Rightarrow y = \frac{c}{x} \Rightarrow xy = c$

Now at $x = 2$, $y = 3$

$\Rightarrow c = 6$

$\therefore xy = 6 \Rightarrow y = \frac{6}{x}$

19. $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$

$\Rightarrow y^2\frac{dx}{dy} + x = \frac{1}{y} \Rightarrow \frac{dx}{dy} + \frac{x}{y^2} + \frac{1}{y^3}$

$\therefore$ Integrating factor (I.F.) $= e^{\int\frac{1}{y^2}dy} = e^{-1/y}$

$\therefore$ General solution is –

$x \,.\, e^{-1/y} = \int\frac{1}{y^3}e^{-1/y}dy + c$

Let $I_1 = \int\frac{1}{y^3}e^{-1/y}dy$

put $\frac{-1}{y} = t$

$y^{-2}\,dy = dt$

$\therefore I_1 = -\int te^t dt$

$= -e^t(t-1)$

$= e^t(1-t)$

$\therefore$ General solution is

$xe^{-1/y} = e^{-1/y}\left(1 + \frac{1}{y}\right) + C$

$\Rightarrow x = 1 + \frac{1}{y} + Ce^{1/y}$

Put $x = 1$, $y = 1$

$\therefore 1 = 1 + \frac{1}{1} + Ce^{1/1}$

$\Rightarrow C = -1/e$

$\therefore x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$

20. $\frac{dP(t)}{dt} = \frac{1}{2}P(t) - 450$

integrate

$\int\frac{dP}{P - 900} = \int\frac{1}{2}dt$

$\ell n|(P - 900)| = \frac{1}{2}t + C$ ....(1)

given $t = 0 \to P = 850$

$\therefore C = \ell n\, 50$

from (1)

$\ln|(P - 900)| = \frac{1}{2}t + \ell n 50$

$\frac{1}{2}t = \ell n\left|\left(\frac{P-900}{50}\right)\right|$

$t = 2\,\ell n\left|\left(\frac{P-900}{50}\right)\right|$

at $P = 0$

$t = 2\ell n\frac{900}{50}$

$t = 2\ell n 18$

21. $P = 100x - 12x^{3/2}.\frac{2}{3} + C$

$x = 0$, $P = 2000$

$C = 2000$

$p_{(x=25)} = 2500 - 1000 + 2000 = 3500$

1. Let X_0 be initial population of the country and Y_0 be its initial food production.

Let the average consumption be a units. Therefore, food required initially aX_0. It is given

$$Y_0 = aX_0\left(\frac{90}{100}\right) = 0.9\ aX_0 \quad \dots (1)$$

Let X be the population of the country in year t.

Then $\frac{dX}{dt}$ = rate of change of population

$$= \frac{3}{100}X = 0.03\ X$$

$$\therefore \frac{dX}{X} = 0.03\ dt$$

Integrating $\int\frac{dX}{X} = \int 0.03dt$

$\Rightarrow$ $\log X = 0.03t + c$

$\Rightarrow$ $X = A\ .\ e^{0.03t}$ where $A = e^c$

At $t = 0$, $X = X_0$, thus $X_0 = A$

$\therefore$ $X = X_0\ e^{0.03t}$

Let Y be the food production in year t.

Then $Y = Y_0\left(1+\frac{4}{100}\right)^t = 0.9aX_0\ (1.04)^t$

($\because$ $Y_0 = 0.9aX_0$ from (1))

Food consumption in the year t is $aX_0\ e^{0.03\ t}$.

Again for no food deficit, $Y - X \geq 0$

$\Rightarrow$ $0.9\ X_0\ a\ (1.04)^t > a\ X_0\ e^{0.03\ t}$

$$\Rightarrow \frac{(1.04)^t}{e^{0.03t}} > \frac{1}{0.9} = \frac{10}{9}$$

Taking log on both sides,

$t[\ell n\ (1.04) - 0.03] \geq \ell n\ 10 - \ell n\ 9$

$$\Rightarrow t \geq \frac{\ell n 10 - \ell n 9}{\ell n(1.04) - 0.03}$$

Thus the least integral values of the year n, when the country becomes self sufficient, is the smallest integer greater than or equal to $\frac{\ell n 10 - \ell n 9}{\ell n(1.04) - 0.03}$

3. $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$ I.F. $= e^{\int\frac{t}{1+t}dt} = e^{-t}(1 + t)$

$\therefore$ solution is $ye^{-t}(1 + t) = \int e^{-t}(1 + t)\frac{1}{1+t} + C$

$ye^{-t}(1 + t) = -e^{-t} + c$ given $y(0) = -1 \Rightarrow c = 0$

$y(1 + t) = -1$ or $y = -\frac{1}{1+t}$

and $y(1) = -\frac{1}{1+1} = -\frac{1}{2}$

5. Given : liquid evaporates at a rate proportional to its surface area.

$$\Rightarrow \frac{dv}{dt} \propto -S \quad \dots (1)$$

We know, volume of liquid $= \frac{1}{3}\pi r^2 h$

and surface area $= \pi r^2$ (of liquid in contact with air)

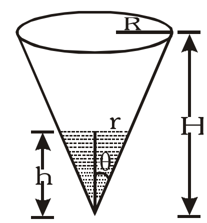
or $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r^2$ ... (2)

Also, $\tan\theta = \frac{R}{H} = \frac{r}{h}$ ... (3)

From (2) and (3),

$$V = \frac{1}{3}\pi r^3 \cot\theta \text{ and } S = \pi r^2 \quad \dots (4)$$

Substituting (4) in (1), we get



$$\frac{1}{3}\pi\cot\theta.\ 3r^2\ .\ \frac{dr}{dt} = -K\pi r^2$$

$$\Rightarrow \cot\theta\int_R^0 dr = -K\int_0^T dt$$

where T is required time after which the cone is empty.

$\Rightarrow$ $\cot\theta\ (0 - R) = -K(T - 0)$

$\Rightarrow$ $R\cot\theta = KT$

$\Rightarrow$ $H = KT$ (using (3))

$\Rightarrow$ $T = \frac{H}{K}$

6. $\frac{dy}{dx} = \frac{-\cos x(1+y)}{2+\sin x}$ or $\int\frac{dy}{1+y} = \int\frac{-\cos x dx}{2+\sin x}$

$\log(1 + y) = -\log(2 + \sin x) + c$

or $\log(1 + y) + \log(2 + \sin x) = c$

Given $y(0) = 1$ means when $x = 0$, $y = 1$

$\Rightarrow$ $\log 2 + \log 2 = c\ \log c \Rightarrow c = 4$

$\Rightarrow$ $1 + y = \frac{4}{2+\sin x}$ or $y = \frac{4}{2+\sin x} - 1$

or $y\left(\frac{\pi}{2}\right) = \frac{4}{3} - 1 = \frac{1}{3}$

8. (a) $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ This is homogeneous so put

$y = vx, \quad \frac{dy}{dx} = v + \frac{dv}{dx}.x$

$v + \frac{dv}{dx}x = \frac{xvx}{x^2+(vx)^2} = \frac{v}{1+v^2}$

or $\quad x\frac{dv}{dx} = \frac{v}{1+v^2} - \frac{v}{1} = \frac{-v^3}{1+v^2}$

$\int -\frac{(1+v^2)dv}{v^3} = \int \frac{dx}{x}$ or $-\int \frac{dv}{v^3} - \int \frac{dv}{v} = \int \frac{dx}{x}$

or $\quad \frac{1}{2v^2} - \log v = \log x + c$

put $v = \frac{y}{x} \Rightarrow \frac{x^2}{2y^2} = \log \frac{y}{x} + \log x + c = \log y + c$

given $y(1) = 1$

$\frac{1}{2} = c \Rightarrow c = 2 \Rightarrow \frac{x^2}{2y^2} = \log y + 2$

Now put $x = x_0, \quad y = e$

$\frac{x_0^2}{2e^2} = 1 + \frac{1}{2} = \frac{3}{2}$ or $x_0^2 = \frac{6e^2}{2} = 3e^2$

$\Rightarrow \quad x_0 = \sqrt{3}.e$

(b) $\frac{xdy - ydx}{y^2} = dy$ or $\frac{ydx - xdy}{y^2} = -dy$

or $\quad \int d\left(\frac{x}{y}\right) = -\int dy$

$\frac{x}{y} = -y + c$ Given $y(1) = 1 \Rightarrow 1 = -1 + c \Rightarrow c = 2$

$\frac{x}{y} = -y + 2$ Now $y(-3)$, $\frac{-3}{y} = -y + 2$

or $\quad y^2 - 2y - 3 = 0$

$y^2 - 3y + y - 3 = 0 = (y-3)(y+1) = 0$

$y = 3$ or $y = -1$ But $y > 0$

$\therefore \quad y = 3$

11. (a) $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

Applying L-hospital, we get

$\Rightarrow \quad x^2 f'(x) - 2x f(x) + 1 = 0$

$\Rightarrow \quad \frac{x^2 f'(x) - 2x f(x)}{(x^2)^2} + \frac{1}{x^4} = 0$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$

$\Rightarrow \quad f(x) = cx^2 + \frac{1}{3x}$

Also $f(1) = 1 \Rightarrow c = \frac{2}{3}$

$\Rightarrow \quad f(x) = \frac{2x^2}{3} + \frac{1}{3x}$

(b) $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y} \Rightarrow \int \frac{y}{\sqrt{1-y^2}}dy = \int dx$

$\Rightarrow \quad -\sqrt{1-y^2} = x + c$

$\Rightarrow \quad (x+c)^2 + y^2 = 1$

Centre $(-c, 0)$; radius = 1

12. $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$

$\sec^{-1}x = \sec^{-1}y + c \qquad \because y(2) = \frac{2}{\sqrt{3}} \ \therefore c = \frac{\pi}{6}$

$\sec^{-1}x = \sec^{-1}y + \frac{\pi}{6} \quad \Rightarrow y = \sec(\sec^{-1}x - \frac{\pi}{6})$

Now $\cos^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y} + \frac{\pi}{6}$

$\cos^{-1}\frac{1}{y} = \cos^{-1}\frac{1}{x} - \cos^{-1}\frac{\sqrt{3}}{2}$

$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{2}\right)$

$\frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$

Hence S(I) is true and S(II) is false.

13. (A) $\frac{dy}{dx} = -\frac{y}{(x-3)^2} \Rightarrow \ln y = \frac{1}{x-3} + c$

$\Rightarrow y = e^{\frac{1}{x-3}+c}, \ x \neq 3.$

(B) $I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$

Applying $x \to 6 - x$

$I = \int_1^5 (5-x)(4-x)(3-x)(2-x)(1-x)\,dx = -I$

$\Rightarrow I = 0.$

(C) $f(x) = \cos^2 x + \sin x$

$f'(x) = -2\cos x \sin x + \cos x$

$\Rightarrow \cos x(-2\sin x + 1) = 0$

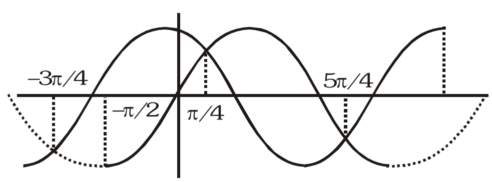
$\cos x = 0$ or $\sin x = \dfrac{1}{2}$

sign of $f'(x)$ changes from –ve to +ve while $f(x)$

passes through $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

(D) $f(x) = \tan^{-1}(\sin x + \cos x)$

$f(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos)^2} > 0$



$x \in (-3\pi/4, \pi/4)$

14. Given $y = f(x)$

Tangent at point $P(x, y)$

$Y - y = \left(\dfrac{dy}{dx}\right)_{(x,y)}(X - x)$

Now y–intercept $\Rightarrow Y = y - x\dfrac{dy}{dx}$

Given that, $y - x\dfrac{dy}{dx} = x^3$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{y}{x} = -x^2$ is a linear differential equation

with I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ell nx} = e^{\ell n\left(\frac{1}{x}\right)} = \dfrac{1}{x}$

Hence, solution is $\dfrac{y}{x} = \int -x^2 . \dfrac{1}{x}dx + C$

or $\dfrac{y}{x} = -\dfrac{x^2}{2} + C$

Given $f(1) = 1$

Substituting we get, $C = \dfrac{3}{2}$

so $y = -\dfrac{x^3}{2} + \dfrac{3}{2}x$

Now $f(-3) = \dfrac{27}{2} - \dfrac{9}{2} = 9$

15. (a) (Bonus)

(Comment : The given relation does not hold for x =1, therefore it is not an identity. Hence there is an error in given question. The correct identity must be-)

$$6\int_1^x f(t)dt = 3xf(x) - x^3 - 5,\ \forall x \geq 1$$

Now applying Newton Leibnitz theorem

$6f(x) = 3xf'(x) - 3x^2 + 3f(x)$

$\Rightarrow 3f(x) = 3xf'(x) - 3x^2$

Let $y = f(x)$

$\Rightarrow x\dfrac{dy}{dx} - y = x^2 \quad \Rightarrow \dfrac{xdy - ydx}{x^2} = dx$

$\Rightarrow \int d\left(\dfrac{y}{x}\right) = \int dx$

$\Rightarrow \dfrac{y}{x} = x + C$ (where C is constant)

$\Rightarrow y = x^2 + Cx$

$\therefore f(x) = x^2 + Cx$

Given $f(1) = 2 \Rightarrow C = 1$

$\therefore f(2) = 2^2 + 2 = 6$

(b) Given $y(0) = 0$, $g(0) = g(2) = 0$

Let $y'(x) + y(x) . g'(x) = g(x) g'(x)$

$\Rightarrow y'(x) + (y(x) - g(x)) g'(x) = 0$

$\Rightarrow \dfrac{y'(x)}{g'(x)} + y(x) = g(x)$

$\Rightarrow \dfrac{dy(x)}{dg(x)} + y(x) = g(x)$

$\Rightarrow$ I.F. $= e^{\int d(g(x))} = e^{g(x)}$

$\Rightarrow y(x).e^{g(x)} = \int e^{g(x)} g(x).dg(x)$

$y(x).e^{g(x)} = g(x).e^{g(x)} - e^{g(x)} + c$

put $x = 0$

$\Rightarrow 0 = 0 - 1 + c \quad \Rightarrow c = 1$

$\Rightarrow y(2) . e^{g(2)} = g(2)e^{g(2)} - e^{g(2)} + 1$

$\Rightarrow y(2) = 0 - e^0 + 1 \quad \Rightarrow y(2) = 0$

16. $\dfrac{dy}{dx} - y\tan x = 2x\sec x$

I.F. $= e^{\int -\tan x\, dx} = \cos x$

$\therefore$ Equation reduces to

$y.\cos x = \int 2x.\sec x.\cos x dx$

$\Rightarrow y \cos x = x^2 + C$

$\because y(0) = 0 \Rightarrow 0 = 0 + C$

$\therefore y \cos x = x^2$

$\Rightarrow y(x) = x^2 \sec x$

$\therefore\ y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{16}\sqrt{2}=\frac{\pi^2}{8\sqrt{2}}$ ($\therefore$ (A) is correct)

$y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{9}.2=\frac{2\pi^2}{9}$ ($\therefore$ (C) is wrong)

Also $y'(x) = 2x \sec x + x^2 \sec x \tan x$

$\Rightarrow y'\left(\frac{\pi}{4}\right)=\frac{\pi}{2}.\sqrt{2}+\frac{\pi^2\sqrt{2}}{16}$ ($\therefore$ (B) is wrong)

and $y'\left(\frac{\pi}{3}\right)=2.\frac{\pi}{3}.2+\frac{\pi^2}{9}.2.\sqrt{3}$

$=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$ ($\therefore$ (D) is correct)

17. $f'(x) - 2f(x) < 0$

Multiply both side by e^{-2x}

$e^{-2x}f'(x)-2e^{-2x}f(x)<0$

$\frac{d}{dx}\left(e^{-2x}f(x)\right)<0$

Now, $g(x) = e^{-2x}f(x)$

$\therefore$ g(x) is a decreasing function.

$x>\frac{1}{2}$

$g(x)<g\left(\frac{1}{2}\right)$

$\Rightarrow\ e^{-2x}f(x)<\frac{1}{e}$

$\Rightarrow\ f(x) < e^{2x-1}$

$\Rightarrow\ \int_{1/2}^{1}f(x)dx<\frac{1}{e}\int_{1/2}^{1}e^{2x}dx$

$=\left[\frac{1}{2e}e^{2x}\right]_{1/2}^{1}=\frac{1}{2e}\left(e^2-e\right)=\frac{1}{2}(e-1)$

$\Rightarrow\ \int_{1/2}^{1}f(x)dx<\frac{e-1}{2}$

obviously $f(x)$ is positive

$\therefore\ \int_{1/2}^{1}f(x)dx>0$

18. $\frac{dy}{dx}=\frac{y}{x}+\sec\frac{y}{x}$

Let $y = vx$

$\frac{dy}{dx}=v+x\frac{dv}{dx}$

$v+\frac{xdv}{dx}=v+\sec v$

$\cos v\, dv=\frac{dx}{x}$

$\sin v = \ell nx + c$

$\sin\left(\frac{y}{x}\right)=\ell nx+c$

$\because$ passing through $\left(1,\frac{\pi}{6}\right)$

$\Rightarrow \sin\frac{\pi}{6}=c\Rightarrow c=\frac{1}{2}$

$\therefore\ \sin\frac{y}{x}=\ell nx+\frac{1}{2}$

Paragraph for Question 19 and 20

19. $e^{-x}(f''(x) - 2f'(x) + f(x)) \geq 1$

$D((f'(x) - f(x))e^{-x}) \geq 1$

$\Rightarrow\ D((f'(x) - f(x)e^{-x}) \geq 0$

$\Rightarrow\ (f'(x) - f(x))e^{-x}$ is an increasing function.

As we know that $e^{-x}f(x)$ has local minima at $x=\frac{1}{4}$

$e^{-x}(f'(x) - f(x)) = 0$ at $x=\frac{1}{4}$

Let $F(x) = e^{-x}(f'(x) - f(x))$

$F(x) < 0$ in $\left(0,\frac{1}{4}\right)$

$e^{-x}(f'(x) - f(x) < 0$ in $\left(0,\frac{1}{4}\right)$

$f'(x) < f(x)$ in $\left(0,\frac{1}{4}\right)$

option C

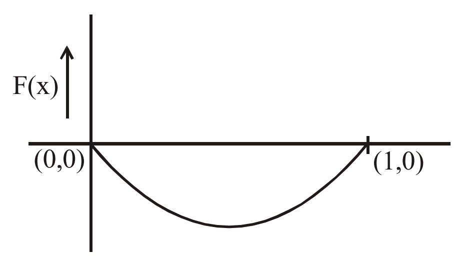
20. $D(e^{-x}(f'(x) - f(x)) \geq 0\ \forall\ x \in (0, 1)$

$D(D(e^{-x}f(x)) \geq 0\ \forall\ x \in (0, 1)$

$D^2(e^{-x}f(x)) \geq 0$

Let $F(x) = e^{-x}f(x)$

$F''(x) > 0$ means it is concave upward.



$F(0) = F(1) = 0$

$F(x) < 0\ \forall\ x \in (0, 1)$

$e^{-x}f(x) < 0\ \forall\ x \in (0, 1)$

$f(x) < 0$

Option D is possible