DEFINITE INTEGRATION

EXERCISE - 01 CHECK YOUR GRASP

4. $I = \int_1^e (x+1)e^x \ell nx\, dx$

$= \int_1^e [x\ell nx + \ell nx + 1 - 1]e^x dx$

$= \int_1^e [\underbrace{x\ell nx}_{f(x)} + \underbrace{\ell nx + 1}_{f'(x)}]e^x dx - \int_1^e e^x dx$

$= [x\ell nx e^x]_1^e - [e^x]_1^e = ee^e - e^e + e$

5. $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx$

$= \int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx +$

$\int_1^2 (1+\cos^8 x)(ax^2+bx+c)\, dx$

$\therefore \int_1^2 (1+\cos^8 x)(ax^2+bx+c)\, dx = 0$

since 1 + $\cos^8$ x is always positive

$= \int_a^b f(x)\, dx = 0$ (b > a)

means f(x) is positive in some portion and negative in some portion from a to b

$\therefore$ ax^2 + bx + c is positive and negative in (1, 2)

$\therefore$ ax^2 + bx + c has a root in (1, 2)

8. Let x = tanθ ⇒ dx = $\sec^2\theta$ dθ

$\therefore I = \int_0^{\pi/2} \frac{\theta\tan\theta}{\sec^2\theta} d\theta = \int_0^{\pi/2} \theta\sin\theta\,\cos\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \theta\sin 2\theta\, d\theta$

$= \frac{1}{2}\left[-\theta\frac{\cos 2\theta}{2} + \int\frac{\cos 2\theta}{2}d\theta\right]_0^{\pi/2} = \frac{\pi}{8}$

11. $\int_2^4 \left[\frac{1}{\log_2 x} - \frac{1}{\ell n2(\log_2 x)^2}\right]dx$

$= \int_2^4 \left[\underbrace{\frac{1}{\log_2 x}}_{f(x)} + \underbrace{\frac{(-x)}{x\ell n2(\log_2 x)^2}}_{f'(x)}\right]dx = \left[\frac{x}{\log_2 x}\right]_2^4 = 0$

12. $F(x) = \int \frac{\sin x}{x}dx$

Now $I = \int_1^3 \frac{\sin 2x}{x}dx$ [put 2x = t]

$= \int_2^6 \frac{2}{2}\frac{\sin t}{t}dt = [F(x)]_2^6 =$ F(6) – F(2)

17. $I = \int_2^3 \frac{(x+2)^2}{2x^2-10x+53}dx$ ......(i)

$I = \int_2^3 \frac{(7-x)^2}{2(5-x)^2-10(5-x)+53}dx$

$I = \int_2^3 \frac{(7-x)^2}{2x^2-10x+53}dx$ .....(ii)

add(i) & (ii)

$2I = \int_2^3 \frac{(x+2)^2+(7-x)^2}{2x^2-10x+53}dx$

$= \int_2^3 dx = 1$ $\therefore I = 1/2$

22. On differentiating both sides

$[f(x)]^2 f'(x) = \cos\pi x - \pi x \sin\pi x$

$[f(9)]^2 f'(9) = -1$ ......(i)

Also $\left[\frac{t^3}{3}\right]_0^{f(x)} = x\cos\pi x \Rightarrow \frac{[f(x)]^3}{3} = x\cos\pi x$

$[f(9)]^3 = -27 \Rightarrow f(9) = -3$ ......(ii)

from (i) & (ii)

$f'(9) = -1/9$

25. $\int_0^{n\pi+V} \sqrt{\frac{2\cos^2 x}{2}}dx = \int_0^{n\pi+V} |\cos x|\, dx$

$= \int_0^{n\pi} |\cos x|\, dx + \int_{n\pi}^{n\pi+V} |\cos x|\, dx$

$= n\int_0^{\pi} |\cos x|\, dx + \int_0^{V} |\cos x|\, dx$

$= 2n + \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{V} \cos x\, dx$ =2n + 2 – sin V

28. $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$ [∵ f(2) = 0]

g(f(x)) = x

g'(f(x))f'(x) = 1

$g'(0)f'(2) = 2$

$$g'(0) = \frac{1}{f'(2)} = \sqrt{17}$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1+x^4}} \Rightarrow f'(2) = \frac{1}{\sqrt{17}}$$

32. $$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{(n+r)(n+2r)}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2\left(1+\frac{r}{n}\right)\left(1+\frac{2r}{n}\right)}$$

$$= \int_0^1 \frac{dx}{(1+x)(1+2x)} = \int_0^1\left(\frac{2}{1+2x} - \frac{1}{1+x}\right)dx$$

$$= \ell n(1+2x) - \ell n(1+x)\Big|_0^1 = \ell n3 - \ell n2 = \ell n\frac{3}{2}$$

34. Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I_1 = I_2 - I_1$

$\therefore \quad 2I_1 = I_2$

hence $\frac{I_1}{I_2} = \frac{1}{2}$

EXERCISE - 02 BRAIN TEASERS

2. $$I_n = \int_0^1 (1+x^2)^{-n}dx$$

$$I_n = [(1+x^2)^{-n}.x]_0^1 + n\int_0^1 \frac{2x^2}{(1+x^2)^{n+1}}dx$$

$$= 2^{-n} + 2n\int_0^1 \frac{(x^2+1)-1}{(x^2+1)^{n+1}}dx$$

$$I_n = 2^{-n} + 2n[I_n - I_{n+1}]$$

$2n\ I_{n+1} = 2^{-n} + (2n - 1)\ I_n$

Now put $n = 1$

$$2I_2 = 2^{-1} + I_1 = \frac{1}{2} + \int_0^1 \frac{1}{1+x^2}dx = \frac{1}{2} + (\tan^{-1}x)_0^1$$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}$$

3. Let $f(x) = \frac{ax^5}{5} + \frac{bx^3}{3} + cx$

It is continuous & differentiable everywhere

Now $f(0) = 0$, $f(1) = \frac{3a+5b+15}{15} = 0$

and $f(-1) = 0$

so $f'(x) = 0$ will have at least one root in $(-1, 0)$ atleast one root in $(0, 1)$, so it will have atleast two roots in $(-1, 1)$

4. $$v = \int_0^\infty \frac{x^2dx}{x^4+7x^2+1}$$

Put $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2}dt$

$$v = -\int_\infty^0 \frac{\frac{1}{t^2}\cdot\frac{1}{t^2}dt}{\frac{1}{t^4}+\frac{7}{t^2}+1} = \int_0^\infty \frac{dx}{x^4+7x^2+1}$$

$v = u$

Hence $$2u = \int_0^\infty\left(\frac{x^2+1}{x^4+7x^2+1}\right)dx$$

$$= \int_0^\infty \left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+7}\right)dx = \int_0^\infty \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+3^2} = \int_0^\infty \frac{dt}{t^2+9}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^\infty$$

$2u = \pi/3$

8. $$I = \int_\infty^0 \frac{ze^{-z}}{\sqrt{1-e^{-2z}}}dz$$

put $e^{-z} = \sin\theta$

$$I = -\int_0^{\pi/2}\frac{\ell n(\sin\theta)(-\cos\theta)d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2}\ell n\sin\theta\, d\theta$$

$$= \frac{-\pi}{2}\ell n2$$

9. $$I = \int_0^{\pi/4}(\cos 2x)^{3/2}\cos x dx$$

$$= \int_0^{\pi/4}(1-2\sin^2 x)^{3/2}\cos x dx$$

put $\sqrt{2}\ \sin x = t \Rightarrow \cos x\ dx = dt/\sqrt{2}$

$$I = \frac{1}{\sqrt{2}}\int_0^1 (1-t^2)^{3/2}dt$$

Now let $t = \sin\theta \Rightarrow dt = \cos\theta\ d\theta$

$$I = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\cos^4\theta d\theta = \frac{3\pi}{16\sqrt{2}}$$

10. $$I = \int_0^1 \prod_{r=1}^{n}(x+r)\sum_{k=1}^{n}\frac{1}{x+K}dx$$

Let $\ell n\prod_{r=1}^{n}(x+r) = t \Rightarrow \sum_{k=1}^{n}\frac{1}{x+K}dx = dt$

$$I = \int_{\ell n(n!)}^{\ell n(n+1)!} e^t dt = [e^t]_{\ell n(n!)}^{\ell n(n+1)!}$$

$= (n + 1)! - n! = n.\ n!$

11. Given $\int_1^2 e^{x^2}dx=\alpha$

Now $I=\int_e^{e^4} 1.\sqrt{\ell n x}\,dx=\left[\left[x\sqrt{\ell n x}\right]_e^{e^4}-\int_e^{e^4}\frac{x}{2x\sqrt{\ell n x}}dx\right]$

Let $I_1=\int_e^{e^4}\frac{dx}{2\sqrt{\ell n x}}$ [Put $x=e^{t^2} \Rightarrow dx=e^{t^2}\,2tdt$]

$=\int_1^2 \frac{e^{t^2}}{2t}.2tdt=\int_1^2 e^{t^2}dt=\alpha$

$\therefore$ $I = 2e^4 - e - \alpha$

14. $I=\int_0^{\pi/2}\frac{\sin x}{x}\frac{\cos x}{\left(\frac{\pi}{2}-x\right)}dx$

$\pi I=\int_0^{\pi/2}\sin 2x\left[\frac{1}{x}+\frac{1}{\pi/2-x}\right]dx$

$=\int_0^{\pi/2}\frac{\sin 2x}{x}dx+\int_0^{\pi/2}\frac{\sin 2x}{\pi/2-x}dx$

$=\int_0^{\pi/2}\frac{\sin 2x}{x}dx+\int_0^{\pi/2}\frac{\sin 2x}{x}dx=4\int_0^{\pi/2}\frac{\sin 2x}{2x}dx$

$\frac{\pi I}{2}=\int_0^{\pi}\frac{\sin t}{t}dt$ [Put $2x = t$]

$I=\frac{2}{\pi}\int_0^{\pi}\frac{\sin x}{x}dx$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

2. Let $I_n=\int_0^1(\ell nx)^n dx$

$=[x(\ell nx)^n]_0^1-n\int_0^1(\ell nx)^{n-1}dx$

$=0-n[x(\ell nx)^{n-1}]_0^1+n(n-1)\int_0^1(\ell nx)^{n-2}dx$

$=(-1)^2n(n-1)\int_0^1(\ell nx)^{n-2}dx$

$I_n = (-1)^n\, n(n-1)$ 1

$=(-1)^n\, n!$

Match the column :

1. **(A)** $I=\int_4^{10}\frac{[x^2]dx}{[(14-x)^2]+[x^2]}$ --------- (i)

$I=\int_4^{10}\frac{[(14-x)^2]}{[x^2]+[(14-x)^2]}dx$ --------- (ii)

add (i) & (ii)

$2I=\int_4^{10}dx$

$\Rightarrow 2I = 6 \Rightarrow I = 3$

(B) $\int_{-1}^{2}\frac{|x|}{x}dx=\int_{-1}^{0}(-1)dx+\int_0^2(1)dx=1$

(C) $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^{99}}{n^{100}}=\int_0^1 x^{99}dx=\left[\frac{x^{100}}{100}\right]_0^1=\frac{1}{100}$

(D) $5050\int_{-1}^{1}\sqrt{x^{200}}dx=5050\times 2\int_0^1|x^{100}|dx$

$=5050\times 2\int_0^1 x^{100}dx$

$=10100\times\left[\frac{x^{101}}{101}\right]_0^1=100=\frac{1}{\alpha}$

$\Rightarrow \alpha=\frac{1}{100}$

Assertion & Reason :

2. Statement-1 :

$I=\int_0^{\pi}x\tan x\cos^3 xdx$ ....(i)

$I=\int_0^{\pi}(\pi-x)\tan x\cos^3 xdx$ ....(ii)

(i) + (ii)

$2I=\pi\int_0^{\pi}\tan x.\cos^3 xdx$

$I=\frac{\pi}{2}\int_0^{\pi}\tan x\cos^3 xdx$ (true)

Statement-2 :

$I=\int_a^b xf(x)dx$ ....(i)

$I=\int_a^b (a+b-x)f(a+b-x)dx$ ....(ii)

(i) + (ii)

$2I=(a+b)\int_a^b f(x)dx$ {If $f(a+b-x)=f(x)$

$$I = \frac{a+b}{2}\int_a^b f(x)dx$$

Hence Statement-2 false

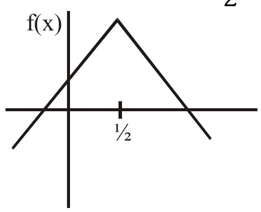
but if $f(a+b-x) \neq f(x)$, then $I \neq \frac{a+b}{2}\int_a^b f(x)$

4. $f(x) = -x^2 + x + 1$

$f'(x) = 1 - 2x$

$f'(x) > 0 \Rightarrow 1 - 2x > 0 \Rightarrow x < \frac{1}{2}$

$f'(x) < 0 \Rightarrow 1 - 2x < 0 \Rightarrow x > \frac{1}{2}$



$\Rightarrow$ f(x) is increasing in $(0, \frac{1}{2})$ and decreasing in $(\frac{1}{2}, 1)$

Now $g(x) = \max\{f(t) ; 0 \le t \le x\}$

$$= \begin{cases} x - x^2 + 1 & 0 \le x \le \frac{1}{2} \\ \frac{5}{4} & \frac{1}{2} \le x \le 1 \end{cases}$$

$$\int_0^1 g(x)dx = \int_0^{1/2}(x - x^2 + 1)dx + \int_{1/2}^1 5/4\,dx = \frac{29}{24}$$

5. $\int_{-\pi}^{\pi}(\sin mx.\sin nx)\,dx = 0$ if $m \neq n$

and $\int_{-\pi}^{\pi}(\sin mx.\sin nx)\,dx = \pi$ if $m = n$

$\therefore$ $a = \cos 0 = 1$ and $b = \cos\pi = -1$

$\therefore$ $a + b = 0$

6. Statement-1 :

Put $x = \frac{1}{t}$ $\Rightarrow$ $dx = -\frac{1}{t^2}dt$

$$I = -\int_3^{1/3} t\,\text{cosec}^{99}\left(\frac{1}{t} - t\right)\frac{1}{t^2}dt$$

$$= -\int_{1/3}^{3} \frac{1}{t}\text{cosec}^{99}\left(t - \frac{1}{t}\right)dt$$

$I = -I$ $\Rightarrow$ $2I = 0$ $\Rightarrow$ $I = 0$

Comprehension # 1 :

1. $g(x) = \int_0^x f(t)dt$

$g'(x) = f(x)$

From the graph it is clear that

$f(x) > 0$ in $x \in [0, 3)$ and

$f(x) < 0$ in $x \in (3, 7)$

$\therefore$ g(x) is increasing in [0, 3] and

g(x) is decreasing in [3, 7]

$\therefore$ maximum value of g(x) occurs at x = 3

$$\therefore\ g(3) = \int_0^3 f(t)dt$$

$$= \int_0^1 1.dt + \int_1^2 (2t-1)dt + \int_2^3 (3t+9)dt$$

$$= 1 + \left(t^2 - t\right)_1^2 + \left(9t - 3\frac{t^2}{2}\right)_2^3$$

$$= 1 + (4 - 2 - 0) + \left(27 - \frac{27}{2} - 18 + 6\right) = \frac{9}{2}$$

2. g(x) start decreasing from x = 3

$$g(4) = \int_0^4 f(t)dt = \int_0^3 f(t)dt + \int_3^4 f(t)dt$$

$$= \frac{9}{2} + \int_3^4 (-3t+9)dt = \frac{9}{2} + \left(9t + \frac{3t^2}{2}\right)_3^4 =$$

$$\frac{9}{2} + \left(36 - 24 - 27 + \frac{27}{2}\right) = 3$$

Now, $g(x) = \int_0^x f(t)dt$

$$= \int_0^4 f(t)dt + \int_4^x f(t)dt \qquad 0 \le x \le 6$$

$$= 3 + \int_4^x (-3)dt = 3 - 3(x-4) = 15 - 3x$$

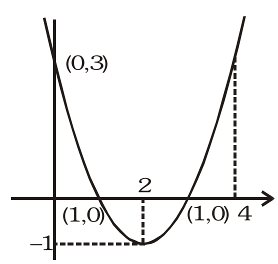
$g(x) = 0$ $\Rightarrow$ $15 - 3x = 0$ $\Rightarrow$ $x = 5$

wich lies in [0, 6]

3. g(x) becomes zero at x = 5

$\therefore$ g(x) will be negative in (5, 7)

Comprehension # 3 :



$f(x) = x^2 - 4x + 3$

$f(x)\big|_{max} = 3$ $x \in [0, 4]$

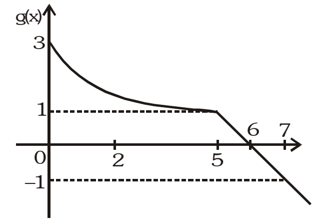
$$f(x)\big|_{min} = \begin{cases} x^2 - 4x + 3 & x \in [0, 2) \\ -1 & x \in [2, 4] \end{cases}$$

$$\text{Now, } g(x) = \begin{cases} \dfrac{x^2-4x+6}{2} & 0 \le x < 2 \\ \dfrac{3-1}{2} = 1 & 2 \le x \le 4 \\ -x+5+x-4 = 1 & 4 < x < 5 \\ \tan\left(\tan^{-1}\left(\dfrac{6-x}{1}\right)\right) = 6-x & x \ge 5 \end{cases}$$

$$g(x) = \begin{cases} \dfrac{x^2-4x+6}{2} & 0 \le x < 2 \\ 1 & 2 \le x < 5 \\ 6-x & x \ge 5 \end{cases}$$

1. $\int_2^5 g(x)\,dx = 5 - 2 = 3$

2. $h(x) = \int_0^{x^2} g(t)dt$



$h'(x) = g(x^2) \,.\, 2x$

$g(x^2) = 0$ at $x = \sqrt{6}$

$\therefore$ $h'(x) < 0$ in $(\sqrt{6}, 7]$

and hence h(x) is decreasing

3. $\lim\limits_{x\to 4} \dfrac{g(x)-g(2)}{\ell n(\cos(4-x))}$ $\quad\left(\dfrac{0}{0} \text{ from}\right)$

$$\lim_{x\to 4} \frac{g'(x)}{\dfrac{1}{\cos(4-x)}\left(\sin(4-x)\right)}$$

$$= \lim_{x\to 4} \frac{g'(x)}{\tan(4-x)} \quad \left(\frac{0}{0} \text{ from}\right)$$

$$\Rightarrow \lim_{x\to 4} \frac{-g''(x)}{\sec^2(4-x)} = 0 \qquad (\because g''(4) = 0)$$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

5. (a) $\int_0^2 [x^2]dx = \int_0^1 0.dx + \int_1^{\sqrt{2}} dx + \int_{\sqrt{2}}^{\sqrt{3}} 2.dx + 3\int_{\sqrt{3}}^2 dx$

$= 5 - \sqrt{2} - \sqrt{3}$

(b) $\int_{-1}^1 [\cos^{-1} x]dx = 3\int_{-1}^{\cos 3} dx + 2\int_{\cos 3}^{\cos 2} dx + \int_{\cos 2}^{\cos 1} dx + \int_{\cos 1}^{0} 0.dx$

$= \cos 1 + \cos 2 + \cos 3 + 3$

8. $\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \dfrac{x^4[(1+x^2)-2x]^2}{1+x^2}dx$

$$= \int_0^1 x^4(1+x^2)dx - 4\int_0^1 x^5 dx + 4\int_0^1 \frac{x^6}{1+x^2}dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{7}\right]_0^1 - 4\left[\frac{x^6}{6}\right]_0^1 + 4\int_0^1 \frac{-dx}{1+x^2} +$$

$$4\int_0^1 \frac{(x^2+1)(x^4+1-x^2)}{1+x^2}dx$$

$$= \left(\frac{1}{5}+\frac{1}{7}\right) - 4\left(\frac{1}{6}\right) - 4\left[\tan^{-1} x\right]_0^1 + 4\left(\frac{1}{5}+1-\frac{1}{3}\right)$$

$$= \frac{22}{7} - \pi$$

9. $I = \int_0^{\pi/2} \dfrac{a\sin x + b\cos x}{\sin\left(\dfrac{\pi}{4}+x\right)}dx$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx \qquad .....(1)$$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x} \qquad .....(2)$$

add.(1) & (2)

$$2I = \sqrt{2}(a+b)\int_0^{\pi/2} dx \;\Rightarrow\; I = \frac{(a+b)\pi}{2\sqrt{2}}$$

12. (a) $I = \int_0^1 \dfrac{1-x}{1+x} \cdot \dfrac{dx}{\sqrt{x+x^2+x^3}}$

$$= \int_0^1 \frac{(1-x^2)}{(x^2+2x+1)\sqrt{x+x^2+x^3}}$$

$$= \int_0^1 \frac{\left(\dfrac{1}{x^2}-1\right)dx}{\left(x+\dfrac{1}{x}+2\right)\sqrt{x+\dfrac{1}{x}+1}}$$

Put $x + \dfrac{1}{x} + 1 = t^2 \Rightarrow \left(1-\dfrac{1}{x^2}\right)dx = 2t\,dt$

$$I = \int_\infty^{\sqrt{3}} \frac{-2t\,dt}{(t^2+1)t} = 2\int_{\sqrt{3}}^\infty \frac{dt}{t^2+1} = 2[\tan^{-1} t]_{\sqrt{3}}^\infty = \frac{\pi}{3}$$

14. $I = \int_0^1 \dfrac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$ ....(i)

$$I = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{(1-x)^2-(1-x)+1}dx = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2-x+1}dx \text{ ...(ii)}$$

Add. (i) and (ii)

$$2I = \int_0^1 \frac{\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{1-x}}{x^2-x+1}dx$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^1 \frac{dx}{(x-1/2)^2 + (\sqrt{3}/2)^2}$$

$$\Rightarrow I = \frac{\pi}{4}\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^1 = \frac{\pi^2}{6\sqrt{3}}$$

16. $I=\int_0^{\pi}\frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ ....(i)

then,

$$I=\int_0^{\pi}\frac{(\pi-x)\sin 2(\pi-x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx \quad ....(ii)$$

$$=\int_0^{\pi}\frac{(\pi-x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{\pi-2x}dx$$

$$=\int_0^{\pi}\frac{(x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

add equation (i) & (ii)

$$2I=\int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\therefore \quad I=\int_0^{\pi}\sin x\cos x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

Put $\frac{\pi}{2}\cos x=t \Rightarrow \sin x dx=-\frac{2}{\pi}dt$

$$\therefore \quad I=-\frac{2}{\pi}\int_{\pi/2}^{-\pi/2}\frac{2t}{\pi}\sin t dt=\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2}t\sin t dt$$

$$\Rightarrow \quad I=\frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2}t\sin t dt=\frac{4}{\pi^2}\left[-t\cos t+\sin t\right]_{-\pi/2}^{\pi/2}$$

$$=\frac{4}{\pi^2}\times 2=\frac{8}{\pi^2}$$

18. $\int_1^2\frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}=\int_1^2\frac{x(x^2-1)dx}{x^4\sqrt{2x^4-2x^2+1}}$

Let $x^2=t \Rightarrow xdx=dt/2$

$$=\frac{1}{2}\int_1^4\frac{(t-1)dt}{t^2\sqrt{2t^2-2t+1}}$$

$$=\frac{1}{2}\int_1^4\frac{t-1}{t^3\sqrt{2-\frac{2}{t}+\frac{1}{t^2}}}dt=\frac{1}{2}\int_1^4\frac{\frac{1}{t^2}-\frac{1}{t^3}}{\sqrt{2-\frac{2}{t}+\frac{1}{t^2}}}dt$$

Let $2-\frac{2}{t}+\frac{1}{t^2}=z^2 \Rightarrow \left(\frac{2}{t^2}-\frac{2}{t^3}\right)dt=2zdz$

$$=\frac{1}{2}\int_1^{5/4}\frac{zdz}{\sqrt{z^2}}=\frac{1}{2}\int_1^{5/4}dz=\frac{1}{8}=\frac{U}{V}$$

$$\Rightarrow (1000)\frac{U}{V}=\frac{1000}{8}=125$$

19. $J_m=\int_1^e \ell n^m x\,dx=[x\ell n^m x]_1^e-m\int_1^e \ell n^{m-1}x.\frac{1}{x}dx$

$=e-mJ_{m-1}$

20. **(c)** $I=\int_0^1\frac{dx}{2+x^2}+\int_1^2\frac{dx}{2+x^2}$

$$\frac{1}{3}\le\int_0^1\frac{dx}{2+x^2}\le\frac{1}{2} \quad ......(1)$$

$$\frac{1}{6}\le\int_1^2\frac{dx}{2+x^2}\le\frac{1}{3} \quad ......(2)$$

add (1) & (2)

$$\frac{1}{2}\le I\le\frac{5}{6}$$

24. **(a)** $f(x)=\int_0^{\sin^2 x}\sin^{-1}\sqrt{t}dt+\int_0^{\cos^2 x}\cos^{-1}\sqrt{t}dt$

Put $t=\sin^2\theta$ in I[st] integral and $t=\cos^2\phi$ in the second integral

then $f(x)=\int_0^x\theta\sin 2\theta d\theta-\int_{\pi/2}^x\phi\sin 2\phi d\phi$

$$=\int_0^x\theta\sin 2\theta d\theta+\int_x^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\int_0^{\pi/2}\theta\sin 2\theta d\theta=\frac{\pi}{4}$$

27. **(c)** Let $P=\lim_{n\to\infty}\left(\frac{\underline{|n}}{n^n}\right)^{1/n}$

$$P=\lim_{n\to\infty}\left(\frac{1.2.3.4.............n}{n.nnn..............n}\right)^{1/n}$$

$$P=\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)........\left(\frac{n}{n}\right)\right)^{1/n}$$

$$\Rightarrow \ell n P=\lim_{n\to\infty}\frac{1}{n}\left(\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+.....\log\left(\frac{n}{n}\right)\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\frac{r}{n}=\int_0^1\ell nx dx=[x\ell nx-x]_0^1$$

$$=(0-1)-\lim_{x\to 0}(x\ell nx)+0$$

$$=-1-\lim_{x\to 0}\frac{\ell nx}{1/x}=-1-\lim_{x\to 0}\frac{1/x}{(-1/x^2)}$$

$$=-1-\lim_{x\to 0}x=-1+0=-1$$

$\Rightarrow \ell np=-1$

$P=e^{-1}=1/e$

1. $$\int_a^b \frac{x^{n-1}\{nx^2-2x^2+n(a+b)x-(a+b)x+nab\}}{(x+a)^2(x+b)^2}dx$$

$$= \int_a^b \frac{x^{n-1}\{n(x+a)(x+b)-x(2x+a+b)\}}{(x+a)^2(x+b)^2}dx$$

$$= \int_a^b \frac{nx^{n-1}}{(x+a)(x+b)}dx - \int_a^b \frac{x^n(x+a+x+b)}{(x+a)^2(x+b)^2}dx$$

$$= \int_a^b \left(\frac{d}{dx}\frac{x^n}{(x+a)(x+b)}\right)dx$$

$$= \left[\frac{x^n}{(x+a)(x+b)}\right]_a^b = \frac{b^{n-1}-a^{n-1}}{2(a+b)}$$

3. $$I = \int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$$

Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx$

$= (-5+4)\int_0^1 e^{((-5+4)x-4+5)^2}dx$

$\{$using property $\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x+a)dx$ $\}$

$$= -\int_0^1 e^{(x-1)^2}dx$$

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right)\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{3}\right)x+\frac{1}{3}-\frac{2}{3}\right]^2}dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2}dx = \frac{-1}{3}I_1$$

where $I = I_1 + 3I_2$

$= I_1 + 3(-I_1/3) = 0$

$\therefore\ I = 0$

6. $x^2 + 2x + 1 = k + 1 + \int_0^1 |t+k|\,dt$

$(x+1)^2 = (k+1) + \int_0^1 |t+k|\,dt$

If $k \geq -1$ R.H.S. ≥ 0

so there will be two real and distinct roots for $k \geq -1$

If $k < -1$

$(x+1)^2 = k + 1 - \int_0^1 (t+k)dt$

$(x+1)^2 = 1/2$

so there will have two real and distinct roots for $k < -1$

$\Rightarrow$ The equation will have two real and distinct roots for $k \in R$,

8. $$I_n = \int_0^1 e^x.(x-1)^n dx$$

$$= e^x.(x-1)^n\Big|_0^1 - n\int_0^1 e^x(x-1)^{n-1}dx$$

$I_n = -(-1)^n - n\,I_{n-1} = (-1)^{n+1} - n\,I_{n-1}$

$n = 1,\ I = \int_0^1 e^x(x-1)dx$

$= (x-1)e^x\Big|_0^1 - \int_0^1 e^x dx = 2 - e$

$I_2 = -1 - 2(2-e) = 2e - 5$

$I_3 = 1 - 3.(2e-5) = 16 - 6e$

so $n = 3$

10. $$f(x) = x + x\int_0^1 y^2 f(y)dy + x^2\int_0^1 yf(y)dy$$

$$= x\left(1+\int_0^1 y^2f(y)dy\right) + x^2\left(\int_0^1 yf(y)dy\right)$$

$\Rightarrow$ f(x) is a quadratic expression of the form $ax + bx^2$

where $a = 1 + \int_0^1 y^2 f(y)dy$

$$= 1 + \int_0^1 y^2(ay+by^2)dy$$

$$a = 1 + \frac{a}{4} + \frac{b}{5}$$

$\Rightarrow 15a - 4b = 20$ ....(i)

and $b = \int_0^1 yf(y)dy = \int_0^1 y(ay+by^2)dy$

$$b = \frac{a}{3} + \frac{b}{4} \Rightarrow 9b - 4a = 0 \quad .....(ii)$$

from (i) and (ii)

$$a = \frac{180}{119}, b = \frac{80}{119}$$

so $f(x) = \dfrac{80x^2 + 180x}{119}$

11. $u_n = \{x(1-x)\}^n$

$$\frac{du_n}{dx} = n\{x(1-x)\}^{n-1}\{1-2x\}$$

$$\frac{du_n}{dx} = n.u_{n-1} - 2nxu_{n-1}$$

$$\frac{d^2u_n}{dx^2} = n(n-1)u_{n-2}\{1-2x\}$$

$$- 2n\{u_{n-1} + x.(n-1)u_{n-2}\{1-2x\}\}$$

$= n(n-1)u_{n-2} - 2xn(n-1)u_{n-2}$

$\quad -2n.u_{n-1} - x2n(n-1)(1-2x)u_{n-2}$

$= n(n-1)u_{n-2} - 2nx(n-1)u_{n-2}\{1+1-2x\} - 2n\,u_{n-1}$

$= n(n-1)u_{n-2} - 4nx(1-x)u_{n-2}\,(n-1) - 2n\,u_{n-1}$

$= n(n-1)u_{n-2} - 2nu_{n-1}\,\{2n-1\}$

$$v_n = \int_0^1 \underset{\text{II}}{e^x}.\underset{\text{I}}{u_n}\,dx$$

& apply by parts twice

13. (a) $\displaystyle\int_0^1 \underset{\text{I}}{x^m}\underset{\text{II}}{(1-x)^n}\,dx$

$$= \left[-x^m\frac{(1-x)^{n+1}}{n+1}\right]_0^1 + \frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1}dx$$

$$= 0 + \frac{m}{n+1}\int_0^1 \underset{\text{I}}{x^{m-1}}\underset{\text{II}}{(1-x)^{n+1}}dx$$

$$= \frac{m(m-1)}{(n+1)(n+2)}\int_0^1 x^{m-2}(1-x)^{n+2}dx$$

....

$$= \frac{m(m-1).........1}{(n+1)(n+2)........(n+m+1)} = \frac{\underline{|m}\,\underline{|n}}{\underline{|m+n+1}}$$

14. $(1-x)^n = C_0 - C_1x + C_2x^2 \;.........+(-1)^nC_nx^n$

$x^{n-1}(1-x)^{n+1} = (C_0x^{n-1} - C_1x^n + C_2x^{n+1} + ...(-1)^n\,C_nx^{2n-1})(1-x)$

$= (C_0x^{n-1} - C_1x^n + C_2x^{n+1} + \;....(-1)^n\,C_nx^{2n-1})$

$- (C_0x^n - C_1x^{n+1} + C_2x^{n+2} + \;........(-1)^n\,C_nx^{2n})$

$$\int_0^1 x^{n-1}(1-x)^{n+1}dx$$

$$= \left[\frac{C_0x^n}{n} - \frac{C_1x^{n+1}}{n+1} + \frac{C_2x^{n+2}}{n+2} - ....\frac{(-1)^nC_nx^{2n}}{2n}\right]_0^1$$

$$-\left[\frac{C_0x^{n+1}}{n+1} - \frac{C_1x^{n+2}}{n+2} + \frac{C_2x^{n+3}}{n+3} - .... + \frac{(-1)^nC_nx^{2n+1}}{2n+1}\right]_0^1$$

$$= \left[\frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} + ...\frac{(-1)^nC_n}{2n}\right]$$

$$-\left(\frac{C_0}{n+1} - \frac{C_1}{n+2} + \frac{C_2}{n+3} + ...(-1)^n\frac{C_n}{2n+1}\right)$$

$$= \frac{C_0}{n(n+1)} - \frac{C_1}{(n+1)(n+2)} + \frac{C_2}{(n+2)(n+3)} + ...$$

upto (n + 1) terms

$$\int_0^1 x^{n-1}(1-x)^{n+1}dx$$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\,\cos\theta\,d\theta$

$$\int_0^1 x^{n-1}(1-x)^{n+1}dx = \int_0^{\pi/2}\sin^{2n-2}\theta\cos^{2n+2}\theta(2\sin\theta\cos\theta)d\theta$$

$$= 2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2n+3}\theta\,d\theta$$

$$= \frac{2\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{2n+3+1}{2}\right)}{2\Gamma\left(\frac{2n-1+2n+3+2}{2}\right)}$$

$$= \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+2)} = \frac{\underline{|n-1}\,\underline{|n+1}}{\underline{|2n+1}}$$

3. $= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{10\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx$

$= \int_0^{10\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx$

$= 10\int_0^{\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx = 9\int_0^{\pi} |\sin x|\,dx$

$= 9 \quad 2 = 18$

4. $I = \int_0^{\sqrt{2}} [x^2]dx = \int_0^{1} [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} dx = [x]_1^{\sqrt{2}} = \sqrt{2} - 1$

11. $f(y) = e^y,\ g(y) = y\ ;\ y > 0$

and $F(t) = \int_0^t f(t-y)g(y)dy$.

$= \int_0^t e^{t-y}y\,dy = e^t\int_0^t e^{-y}y\,dy = e^t[-ye^{-y} - e^{-y}]_0^t$

$= -e^t[te^{-t} + e^{-t} - 0 - 1] = e^t - (1 + t)$

17. $f(x) = \frac{e^x}{1+e^x}$ $\qquad I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)]dx$

$I_2 = \int_{f(-a)}^{f(a)} g[x(1-x)]dx$

$f(a) = \frac{e^a}{1+e^a}$, $f(-a) = \frac{e^{-a}}{1+e^{-a}}$

$\therefore f(a) + f(-a) = 1$

$2I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx + \int_{f(-a)}^{f(a)} \{f(a)+f(-a)-x\}g(1-x)(x)dx$

$2I_1 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx = I_2$

$\because f(a) + f(-a) = 1$

$2I_1 = I_2$

$\frac{I_2}{I_1} = 2$

18. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\sec^2\frac{r^2}{n^2}$

$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}.\frac{r}{n}\sec^2\frac{r^2}{n^2}$ $\qquad$ Put $\frac{1}{n} = dx$; $\frac{r}{n} = x$

lower limit $x = \frac{r}{n}$

$r = 1 \qquad x = 1/n$

$n \to \infty \qquad x = 0$

$r = n \qquad x = 1$

$= \int_0^1 x\sec^2 x^2 dx$

Put $x^2 = t$; $2xdx = dt$; $xdx = \frac{dt}{2}$

$x = 0,\ t = 0$

$x = 1,\ t = 1$

$= \frac{1}{2}\int_0^1 \sec^2 t\,dt$

$= \frac{1}{2}(\tan t)_0^1 \qquad = \frac{1}{2}\tan 1$

19. for $0 < x < 1$, $\qquad x^2 > x^3$ and

for $1 < x < 2$, $\qquad x^3 > x^2$

$\therefore$ for, $0 < x < 1$, $\qquad 2^{x^2} > 2^{x^3}$ and

for $1 < x < 2$, $\qquad 2^{x^2} < 2^{x^3}$

$\therefore \int_0^1 2^{x^2}dx > \int_0^1 2^{x^3}dx$ and $\int_1^2 2^{x^2}dx < \int_1^2 2^{x^3}dx$

$\therefore I_1 > I_2$ and $I_3 < I_4$

21. Putting $\quad - x$ for x

$I = \int_{\pi}^{-\pi}\frac{\cos^2 x}{1+a^{-x}}(-dx) = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^{-x}}dx$

$I + I = \int_{-\pi}^{\pi}\cos^2 x\left(\frac{1}{1+a^x}+\frac{1}{1+a^{-x}}\right)dx$

$= \int_{-\pi}^{\pi}\cos^2 x\,dx \Rightarrow 2I = 2\int_0^{\pi}\cos^2 x\,dx$

$= \int_0^{\pi}(1+\cos 2x)dx$

$2I = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

$2I = \pi \Rightarrow I = \frac{\pi}{2}$

25. $= \int_1^2 1.f'(x)dx + \int_2^3 2.f'(x)dx + ... + \int_{[a]}^a [a]f'(x)dx$

$= [f(2) - f(1)] + 2[f(3) - f(2)] + ... + [a]\ [f(a)]$

$= [a]\ f(a) - \{f(1) + f(2) + ... + f[a]\}$

26. $F(x) = f(x)+f(1/x)$ put $x = e$

$F(e) = \int_1^e \frac{\log t}{1+t}dt + \int_1^{1/e} \frac{\log t}{1+t}dt$

let $t = \frac{1}{z} \Rightarrow \frac{dt}{dz} = \left(\frac{-1}{z^2}\right)$

$= \int_1^e \frac{\ln t}{(1+t)}dt + \int_1^e \frac{\ln 1/z}{(1+1/z)}\left(\frac{-1}{z^2}\right)dz$

by property $\int_a^b f(x)dx = \int_a^b f(t)dt$

$\int_1^e \frac{\ln t}{(1+t)}dt + \int_1^e \frac{\ln t}{t(1+t)}dt = \int_1^e \frac{\ln t}{t}dt = \frac{1}{2}$

28. Now

$\sin x < x \Rightarrow \frac{\sin x}{\sqrt{x}} < \sqrt{x}$

$\int_0^1 \frac{\sin x}{\sqrt{x}}dx < \int_0^1 \sqrt{x}\ dx$

$I < \left[\frac{2}{3}x^{3/2}\right]_0^1$

$I < \frac{2}{3}$

$\because \quad \cos x < 1$

$\frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}}dx < \int_0^1 \frac{1}{\sqrt{x}}dx < \left[2\sqrt{x}\right]_0^1 < 2$

$J < 2$

29. $I = \int_0^\pi [\cot x]dx$ .... (1)

$I = \int_0^\pi [\cot(\pi - x)]dx = \int_0^\pi [-\cot x]dx$ ...... (2)

add (1) & (2)

$2I = \int_0^\pi [\cot x] + [-\cot x]dx \qquad \because [x] + [-x] = -1$

$= \int_0^\pi -1dx = -[x]_0^\pi \Rightarrow I = -\frac{\pi}{2}$

32. $\int_0^{1.5} x[x^2]dx$

$\int_0^1 0\,dx + \int_1^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^{1.5} 2x\,dx$

$\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[x^2\right]_{\sqrt{2}}^{1.5}$

$\left(\frac{2}{2} - \frac{1}{2}\right) + (2.25 - 2)$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

33. $g(x) = \int_0^x \cos 4t\,dt$

$g(x + \pi) = \int_0^{x+\pi} \cos 4t\,dt$

$= \int_0^x \cos 4t\,dt + \int_x^{x+\pi} \cos 4t\,dt$

$= \int_0^x \cos 4t\,dt + \int_0^\pi \cos 4t\,dt = g(x) + g(\pi)$

Because $g(\pi) = 0$ so $g(x) - g(\pi)$ is also correct Ans.

34. Statement-I : $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$I = \int_{\pi/6}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ......(1)

use $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}dx}{\sqrt{\cos x}+\sqrt{\sin x}}$ .....(2)

(1) + (2)

$2I = \int_{\pi/6}^{\pi/3} dx$

$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$I = \frac{\pi}{12}$

So Statement-I is false.

and statement-II is true as it is property.

6. Given that f(x) is an even function, then to prove

$$\int_0^{\pi/2} f(\cos 2x)\cos x dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$$

Let $I = \int_0^{\pi/2} f(\cos 2x)\cos x\, dx$ ....(1)

$$= \int_0^{\pi/2} f\left[\cos 2\left(\frac{\pi}{2}-x\right)\right]\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{Using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} f(-\cos 2x)\sin x dx$$

$I = \int_0^{\pi/2} f(\cos 2x)\sin x dx$ .....(2)

[As $f(x)$ is an even function]

adding two values of I in (1) and (2) we get

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)dx$$

$$\Rightarrow I = \frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]dx$$

$$I = \frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\cos(x-\pi/4)dx$$

Let $x - \pi/4 = t \Rightarrow dx = dt$

$$\therefore\ I = \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[\cos 2(t+\pi/4)]\cos t\, dx$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[-\sin 2t]\cos t\, dt$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\ dt$$

[$\because$ f is an even function]

$$= \frac{2}{\sqrt{2}}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$$

[$\because$ f is an even function]

$$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx = \text{R.H.S.}$$

8. (b) $I = \int_{-2}^{0}[x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]dx$

$$= \left[\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + 3x + (x+1)\sin(x+1) + \cos(x+1)\right]_{-2}^{0}$$

$= 4$

9. $\text{Let} I = \int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\ dx$

$$= \int_0^{\pi} e^{|\cos x|} 2\sin\left(\frac{1}{2}\cos x\right)\sin x dx$$

$$+ \int_0^{\pi} e^{|\cos x|} 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

$= I_1 + I_2$

Now using the property that

$\int_0^{2a} f(x)dx = 0$ if $f(2a - x) = -f(x)$

$= 2\int_0^{a} f(x)dx$ if $f(2a - x) = f(x)$

We get, $I_1 = 0$

and $I_2 = 2\int_0^{\pi/2} e^{|\cos x|} 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$

$$= 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

Put $\cos x = t \Rightarrow -\sin x\ dx = dt$, we get

or $I_2 = 6\int_0^1 e^t\cos\frac{t}{2}dt$

$$I_2 = 6[(e^t\cos\frac{t}{2})_0^1 + \frac{1}{2}\int_0^1 e^t\sin\frac{t}{2}dt]$$

$$= 6\left[e\cos(1/2) - 1 + \frac{1}{2}\left\{(e^t\sin t/2)_0^1 - \frac{1}{2}\int_0^1 e^t\cos t/2 dt\right\}\right]$$

$$I_2 = 6\left[e\cos\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\left\{e\sin(1/2) - \frac{1}{2}.\frac{1}{6}I_2\right\}\right]$$

$$I_2 + \frac{1}{4}I_2 = 6\left[e\cos(1/2) + \frac{1}{2}e\sin(1/2) - 1\right]$$

$$\Rightarrow I_2 = \frac{24}{5}\left[e\cos(1/2) + \frac{1}{2}e\sin\left(\frac{1}{2}\right) - 1\right]$$

10. $\int_0^{\pi/2}\sin x dx = \frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

11. $f''(x) < 0,\ \forall\ x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$$

$$= \frac{1}{2}[(b-a)f'(c) + f(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a)f''(c) < 0$$

$[\because f''(x) < 0, \forall x \in (a,b) \text{ and } b > a]$

$\therefore$ F(c) is max. at the point (c, f(c)) where

$$F'(c) = 0 \Rightarrow f'(c) = \left(\frac{f(b)-f(a)}{b-a}\right)$$

12. $$\lim_{x\to a}\frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$$

$$\lim_{h\to 0}\frac{\int_a^{a+h} f(x)dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a+h) - \frac{1}{2}[f(a)+f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$$

[Using L'Hospital rule]

$$\Rightarrow \lim_{h\to 0}\frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L' Hospital rule]

$$\Rightarrow \lim_{h\to 0}\frac{-f''(a+h)}{12} = 0 \Rightarrow f''(x) = 0,\ \forall\ a \in R$$

$\Rightarrow$ f(x) must be of max. degree 1

13. Let $I = \int_0^1 (1-x^{50})^{100}dx$ and $I' = \int_0^1 (1-x^{50})^{101}dx$

Then, $I' = \int_0^1 1.(1-x^{50})^{101}dx = (x(1-x^{50})^{101})_0^1$

$$+101\int_0^1 50x^{50}(1-x^{50})^{100}dx$$

$$= 5050\int_0^1 x^{50}(1-x^{50})^{100}dx$$

$$-I' = 5050\int_0^1 -x^{50}(1-x^{50})^{100}dx$$

$$\Rightarrow 5050I - I' = 5050\int_0^1 (1-x^{50})^{100}dx$$

$$+5050\int_0^1 -x^{50}(1-x^{50})^{100}dx$$

$$\Rightarrow 5050\int_0^1 (1-x^{50})^{101}dx = 5050\ I'$$

$$\Rightarrow 5050\ I = 5051\ I' \Rightarrow 5050\frac{I}{I'} = 5051$$

17. $$S_n = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n + k^2/n^2}$$

$$S_n < \int_0^1 \frac{dx}{x^2+x+1}$$

($\because$ the function is decreasing)

$$S_n < \int_0^1 \frac{dx}{\left(x+1/2\right)^2 + \left(\sqrt{3}/2\right)^2}$$

$$S_n < \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1$$

$$S_n < \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$S_n < \frac{\pi}{3\sqrt{3}}$$

Now $T_n - S_n = 1 - \frac{1}{3n} \Rightarrow T_n - S_n > \frac{2}{3}$

$\Rightarrow T_n > S_n + \frac{2}{3}$

as $S_n < \frac{\pi}{3\sqrt{3}}$ so $T_n > \frac{\pi}{3\sqrt{3}}$

18. $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt$, $0 \le x \le 1$

differentiating both the sides & squreing

$\Rightarrow \quad 1 - (f'(x))^2 = f^2(x) \Rightarrow \dfrac{f'(x)}{\sqrt{1-f^2(x)}} = 1$

$\Rightarrow \quad \sin^{-1} f(x) = x + c$

$f(0) = 0$

$\Rightarrow \quad f(x) = \sin x \Rightarrow \because \sin x \le x$ for $x \in [0, 1]$

$\Rightarrow \quad f\left(\dfrac{1}{2}\right) < \dfrac{1}{2}$ and $f\left(\dfrac{1}{3}\right) < \dfrac{1}{3}$.

19. $I_n = \int_{-\pi}^{\pi} \dfrac{\sin nx}{(1+\pi^x)\sin x}dx$

$I_n = \int_{-\pi}^{\pi} \dfrac{\pi^x \sin nx}{(1+\pi^x)\sin x}dx$

$2I_n = \int_{-\pi}^{\pi} \dfrac{\sin nx}{\sin x}dx$ ..(i)

$2\,I_{n+2} = \int_{-\pi}^{\pi} \dfrac{\sin(n+2)x}{\sin x}dx$ ...(i)

(ii) – (i)

$\Rightarrow 2(I_{n+2} - I_n) = \int_{-\pi}^{\pi} \cos(n+1)x = 0 \quad \Rightarrow \quad I_{n+2} = I_n$

$\sum_{m=1}^{10} I_{2m} = 10 \sum_{m=1}^{10} I_2 = \dfrac{10}{2}\int_{-\pi}^{\pi} \dfrac{\sin 2x}{\sin x}dx = 0$

Put n = 1 in equation (i)

$2I_1 = \int_{-\pi}^{\pi} \dfrac{\sin x dx}{\sin x} = 2\pi$

$I_1 = \pi$

$\sum_{m=1}^{10} I_{2m+1} = 10\pi$

20. $f(x) = \int_0^x f(t)\,dt$ ...(i)

$f'(x) = f(x) \Rightarrow f(x) = k.e^x$

From (i) $f(0) = 0$

$\Rightarrow f(0) = k.e^0 \Rightarrow k = 0 \Rightarrow f(x) = 0$

21. Applying L-Hospital rule,

$$\lim_{x\to 0} \frac{\int_0^x \frac{t\ell n(1+t)}{t^4+4}dt}{x^3} = \lim_{x\to 0}\frac{\frac{x\ell n(1+x)}{x^4+4}}{3x^2}$$

$$= \lim_{x\to 0}\frac{\ell n(1+x)}{3x(x^4+4)} = \frac{1}{12}$$

22. $I = \int_0^1 \dfrac{x^4(1-2x+x^2)^2}{1+x^2}dx$

$I = \int_0^1 \dfrac{x^4\left\{(1+x^2)^2 - 4x(1+x^2)+4x^2\right\}}{1+x^2}dx$

$= \int_0^1 (1+x^2)x^4dx - \int_0^1 4x^5dx + 4\int_0^1 \dfrac{(x^6+1)-1}{1+x^2}dx$

$= \dfrac{1}{5} + \dfrac{1}{7} - 4.\dfrac{1}{6} + 4\int_0^1 \dfrac{(x^2+1)^3 - 3x^2(1+x^2)}{1+x^2}dx - 4\int_0^1 \dfrac{dx}{1+x^2}$

$= \dfrac{12}{35} - \dfrac{2}{3} + 4\int_0^1 (x^4+2x^2+1)dx - 12\int_0^1 x^2dx - \pi$

$= \dfrac{12}{35} - \dfrac{2}{3} + 4\left(\dfrac{1}{5}+\dfrac{2}{3}+1\right) - 4 - \pi$

$= \dfrac{12}{35} - \dfrac{2}{3} + \dfrac{52}{15} - \pi = \dfrac{22}{7} - \pi$

23. $f(x) = \begin{cases} \{x\} & \text{when } -9 \le x < -8;\ -7 \le x < -6, \ldots\ldots \\ 1-\{x\} & \text{when } -10 \le x \le -9;\ -8 \le x < -7, \ldots\ldots \end{cases}$

Since f(x) & cosπx both are periodic functions having period 2.

$I = \dfrac{10\times\pi^2}{10}\left(\int_0^1 (1-\{x\})\cos\pi x dx + \int_1^2 \{x\}\cos\pi x dx\right)$

$= \pi^2\left(\int_0^1 (1-x)\cos\pi x dx + \int_1^2 (x-1)\cos\pi x dx\right)$

$= \pi^2\left(\int_0^1 \cos\pi x dx - \int_1^2 \cos\pi x dx + \int_1^2 x\cos\pi x dx - \int_0^1 x\cos\pi x dx\right)$

$\Rightarrow \quad I = 4$

24. $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$

$e^{-x}f'(x) - e^{-x}f(x) = \sqrt{x^4+1}$

$\Rightarrow \quad f'(x) - f(x) = e^x\sqrt{x^4+1}$

$\Rightarrow \quad \dfrac{dy}{dx} = y + e^x\sqrt{x^4+1}$ (say) ....(i)

considering y = f(x). so that x = f'(y)

$f^{-1}{}'(2) = \left(\dfrac{dx}{dy}\right)_{y=2}$ ...... (ii)

for x = 0 $\quad \Rightarrow \quad$ f(x) = 2 i.e. y = 2

$\Rightarrow \quad f^{-1}(2) = 0$

$\dfrac{dy}{dx} = 2 + 1\sqrt{1} = 3$

from (2), $f^{-1}{}'(2) = \dfrac{1}{3}$

25. $I = \int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ell n6 - x^2)} dx$; put $x^2 = t$

$\Rightarrow 2xdx = dt$

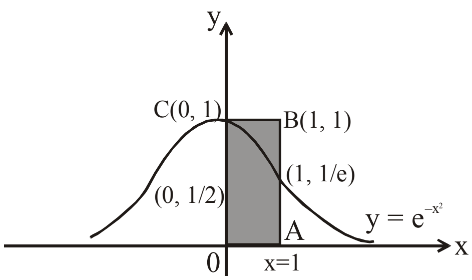
$\Rightarrow \quad I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin t}{\sin t + \sin(\ell n6 - t)} dt \qquad ....(i)$

$\Rightarrow \quad I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin(\ell n6 - t)}{\sin(\ell n6 - t) + \sin t} dt \qquad ....(ii)$

Adding equation (i) & (ii)

$\Rightarrow \quad 2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} dt \qquad \Rightarrow \qquad I = \frac{1}{4}\ell n\left(\frac{3}{2}\right)$

26. Area (OABC) = 1



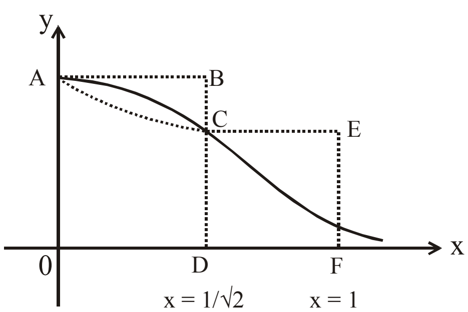
Shaded area is S.

Clearly S < 1

and $\int_0^1 e^{-x^2} dx > \int_0^1 e^{-x} dx$

$\Rightarrow \quad S > 1 - \frac{1}{e}$ ($\therefore$ (B) is correct

Again S $\geq$ Area (trapezium ACDO)



$\Rightarrow \quad S \geq \frac{1}{2}\left(1 + \frac{1}{\sqrt{e}}\right)\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad S \geq \frac{1}{2\sqrt{2}}\left(1 + \frac{1}{\sqrt{e}}\right)$

$\therefore \quad$ C is wrong

Also S $\leq$ Sum of areas of rectangles ABDO & CEFD

$\Rightarrow \quad S \leq \frac{1}{\sqrt{2}} \times 1 + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$

$\Rightarrow \quad S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

($\therefore$ (D) is correct)

27. $\int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx + \int_{-\pi/2}^{\pi/2} \ell n\left(\frac{\pi + x}{\pi - x}\right) \cos x dx$

$= \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx = 2\int_0^{\pi/2} \underset{\mathrm{I}}{x^2} \underset{\mathrm{II}}{\cos x} \, dx$

$= 2\left((x^2 \sin x)_0^{\pi/2} - 2\int_0^{\pi/2} x \sin x dx\right)$

$= 2\left(\frac{\pi^2}{4} - 2\left(-(x\cos x)_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx\right)\right)$

$= 2\left(\frac{\pi^2}{4} - 2\int_0^{\pi/2} \cos x \, dx\right)$

$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$

28. $L = \lim_{n\to\infty} \frac{1^a + 2^a + ... + n^a}{(n+1)^{a-1}\left[\underbrace{na + na + na + ... + na}_{\text{n times}} + 1 + 2 + 3 + ... + n\right]}$

$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]}$

$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)n^{a+1}}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]}$

$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2 a + \frac{n(n+1)}{2}}{n^2}\right]}$

$= \frac{\int_0^1 x^a dx}{\left(a + \frac{1}{2}\right)} = \frac{1}{60} \Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60}$

$\Rightarrow 2a^2 + 3a - 119 = 0 \quad \Rightarrow \quad a = 7 \ \& \ -\frac{17}{2}$

$a = -\frac{17}{2}$ will be rejected as $\int_0^1 x^{-\frac{17}{2}} dx$ is not defined.