CONTINUITY

EXERCISE - 01 — CHECK YOUR GRASP

2. Hint : $\lim_{x\to 0^+} f(x) = 0$ & $\lim_{x\to 0^-} f(x) = 1$

7. $\lim_{x\to 0^-} \dfrac{1-\cos 4x}{x^2} = 8$

$\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} = 8$ $\quad\therefore\quad$ f(0) = 8

so f(x) is continuous at x = 0 when a = 8

11. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$

on rationalizing both Nr. & Dr. we get

$\lim_{x\to 0} f(x) = -\sqrt{a}$ $\quad$ so $\quad$ $f(0) = -\sqrt{a}$

13. $f(x)=\lim_{x\to 0} \dfrac{x\left(1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}...\right)\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}...\right)}{x^3}$

$= \lim_{x\to 0} \dfrac{(1+a-b)+x^2\left(\frac{-a}{2!}+\frac{b}{3!}\right)+\ldots}{x^2}$

$\Rightarrow$ 1 + a – b = 0 ... (i)

and $\dfrac{-a}{2} + \dfrac{b}{6} = 1$ ... (ii)

solving (i) and (ii) we get

$a = \dfrac{-5}{2}$, $b = \dfrac{-3}{2}$

14. $(x-\sqrt{3})\,f(x) = -x^2 + 2x - 2\sqrt{3} + 3$

$f(x) = \dfrac{-x^2+2x-2\sqrt{3}+3}{x-\sqrt{3}}$

$= \dfrac{(x-\sqrt{3})(2-\sqrt{3}-x)}{x-\sqrt{3}} = 2-\sqrt{3}-x$

$f(\sqrt{3}) = 2-2\sqrt{3}$

17. (A) LHL = –1 & RHL = 0

(B) LHL = 1 & RHL = 2/3

(C) LHL = –1 & RHL = 2/3

(D) LHL = $-2\log_2 3$ & RHL = $2\log_3 2$

EXERCISE - 02 — BRAIN TEASERS

3. (i) $\tan f(x) = \tan\left(\dfrac{x}{2}-1\right)$ $\quad x \in [0, \pi]$

$0 \le x \le \pi \Rightarrow -1 \le \dfrac{x}{2}-1 \le \dfrac{\pi}{2}-1$

By graph we say tan(f(x)) is continuous in [0, π]

(ii) $\dfrac{1}{f(x)} = \dfrac{2}{x-2}$ is not defined at $x = 2 \in [0, \pi]$

(iii) $y = \dfrac{x-2}{2}$

$\Rightarrow$ $f^{-1}(x) = 2x + 2$ is continuous in R.

7. $\lim_{x\to 0^+} (x+1)\,e^{-[2/x]} = \lim_{x\to 0^+} \dfrac{x+1}{e^{2/x}} = \dfrac{1}{e^{\infty}} = 0$

$\lim_{x\to 0^-} (x+1)\,e^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} = 1$

Hence continuous for $x \in I - \{0\}$

9. $\text{RHL} = \lim_{x\to 0^+} \left(3-\left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right]\right)$

$=3-[\cot^{-1}(-\infty)] = 3 - 3 = 0$

$\text{LHL} = \lim_{h\to 0} \{(0-h)^2\} \cos\left(e^{\left(\frac{1}{0-h}\right)}\right)$

$= \lim_{h\to 0} (0-h)^2 \cos(e^{-\infty}) = 0$

11. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} b([x]^2 + [x]) + 1$

$=\lim_{h\to 0} b([-1+h]^2 + [-1+h]) + 1$

$= b((-1)^2 - 1) + 1 = 1$

$\Rightarrow b \in R$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \sin(\pi(x+a))$

$= \lim_{h\to 0} \sin(\pi(-1-h+a)) = -\sin \pi a$

$\sin \pi a = -1$

$\pi a = 2n\pi + \dfrac{3\pi}{2} \Rightarrow a = 2n + \dfrac{3}{2}$

Also option (C) is subset of option (A)

12. $\text{LHL} = \lim_{h\to 0} (0-h)\,[0-h]^2 \log_{(1+0-h)} 2$

$= \lim_{h\to 0} \dfrac{-h(-1)^2 \ell n 2}{\ell n(1-h)} = \ell n\, 2$

$\text{RHL} = \lim_{h\to 0} \dfrac{\ell n(e^{(0+h)^2}+2\sqrt{\{0+h\}})}{\tan\sqrt{(0+h)}}$

$= \lim_{h\to 0} \dfrac{\ell n(e^{h^2}+2\sqrt{h})}{\tan\sqrt{h}} = \lim_{h\to 0} \dfrac{\ell n(e^{h^2}+2\sqrt{h})}{\sqrt{h}}$

$$= \lim_{h\to 0}\left[h^2 + \frac{\ell n\left(1+\frac{2\sqrt{h}}{e^{h^2}}\right).\frac{2\sqrt{h}}{e^{h^2}}}{\frac{2\sqrt{h}}{e^{h^2}}}\right]\frac{1}{\sqrt{h}} = 2$$

$\therefore$ RHL $\neq$ LHL

13. $f(0^+) = \lim_{h\to 0} a(\sin^2 (0 + h))^n = 0$

$f(0) = 0$

$f(0^-) = \lim_{h\to 0} b(\cos^2 (0 - h))^m - 1 = -1$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

True & False :

8. $f\left(\frac{1}{\sqrt{2}}^+\right) = f\left(\frac{1}{\sqrt{2}}^-\right) = 1$ and $f(x)$ is continuous in $[0, 1]$. Hence $f\left(\frac{1}{\sqrt{2}}\right)$ will also be 1.

Match the Column :

1. (A) $\lim_{h\to 0} \sin \{1 - h\} = \cos 1 + a$

$\Rightarrow \lim_{h\to 0} \sin (1 - h) - \cos 1 = a$

$\Rightarrow a = \sin 1 - \cos 1$

Now $|k| = \dfrac{\sin 1 - \cos 1}{\sqrt{2}\left(\sin 1.\frac{1}{\sqrt{2}} - \cos 1.\frac{1}{\sqrt{2}}\right)} = 1$

$k = \pm 1$

(B) $f(0) = \lim_{x\to 0} \dfrac{2\sin^2\left(\frac{\sin x}{2}\right)}{x^2\left(\frac{\sin x}{2}\right)^2} \times \left(\frac{\sin x}{2}\right)^2$

$\Rightarrow f(0) = \dfrac{1}{2}$

(C) function should have same rule for Q & Q'

$\Rightarrow x = 1 - x \Rightarrow x = \dfrac{1}{2}$

(D) $f(x) = x + \{-x\} + [x]$

x is continuous at $x \in R$

Check at $x = I$ (where I is integer),

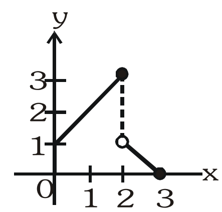
$f(I^+) = 2I + 1$

$f(I^-) = 2I - 1$

So $f(x)$ is discontinuous at every integer i.e., 1,0,–1

Comprehension # 2 :

$$f(x) = \begin{cases} x+1 & 0 \le x \le 2 \\ -x+3 & 2 < x < 3 \end{cases}$$



1. $f(x)$ is discontinuous at $x = 2$

2. $fof(x) = \begin{cases} x+2 & 0 \le x \le 1 \\ -x+2 & 1 < x \le 2 \\ -x+4 & 2 < x < 3 \end{cases}$

$fof(x)$ is discontinuous at $x = 1, 2$

3. $f(19) = f(3 \quad 6 + 1) = f(1) = 2$

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

3. $\lim_{x\to 0^-} \dfrac{\sin(a+1)x + \sin x}{x} = a + 2$

and $\lim_{x\to 0^+} \dfrac{x + bx^2 - x}{bx^{3/2}(\sqrt{x+bx^2}+\sqrt{x})} = \dfrac{1}{2}$ as $b \neq 0$

according to question

$c = \dfrac{1}{2}$ & $a + 2 = \dfrac{1}{2} \Rightarrow a = \dfrac{-3}{2}$

4. $f(0^-) = \lim_{h\to 0}\left(-\dfrac{2^{-1/h}-1}{2^{-1/h}+1}\right) = 1$

$f(0^+) = \lim_{h\to 0}\left(-\dfrac{2^{1/h}-1}{2^{1/h}+1}\right) = -1$

$\Rightarrow$ LHL $\neq$ RHL $\Rightarrow$ Non removable-finite discontinuity

7. $f(x) = \lim_{x\to 0} \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$

$= \lim_{x\to 0} \dfrac{\sin x}{x}\left(\dfrac{3 - 4\sin^2 x + 2A\cos x + B}{x^4}\right)$

$= \lim_{x\to 0} \dfrac{1 + 2\cos 2x + 2A\cos x + B}{x^4}$

$= \lim_{x\to 0} \dfrac{1}{x^4}\left(1 + 2\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \ldots\right) + 2A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + B\right)$

$= \lim_{x\to 0} \dfrac{1}{x^4}\left(3 + 2A + B + x^2(-4-A) + x^4\left(\frac{4}{3} + \frac{A}{12}\right) + \ldots\right)$

$\Rightarrow 2A + B + 3 = 0$ and $-4 - A = 0$

$\Rightarrow A = -4,\ B = 5$

and $f(0) = 1$

8. (b) Given $\lim_{h\to 0} f(1 + h) = \lim_{h\to 0} f(1 - h) = f(1) \neq 0$

Let $x = a \in R - \{0\}$

$f(a\ .\ 1) = f(a)\ f(1) \quad \Rightarrow \quad f(1) = 1$

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f\left(a\left(1+\frac{h}{a}\right)\right)$$

$$= \lim_{h\to 0} f(a)f\left(1+\frac{h}{a}\right) \qquad [\because f(x.y) = f(x).f(y)]$$

Similarly $f(a-h) = f(a)$

Hence $f(x)$ is continuous at $x = R - \{0\}$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. $$f(x) = \sum_{r=1}^{n} \frac{\sin\left(\frac{x}{2^r}\right)}{\cos\left(\frac{x}{2^r}\right)\cos\left(\frac{x}{2^{r-1}}\right)} = \sum_{r=1}^{n} \frac{\sin\left(\frac{x}{2^{r-1}}-\frac{x}{2^r}\right)}{\cos\left(\frac{x}{2^r}\right)\cos\left(\frac{x}{2^{r-1}}\right)}$$

$$= \sum_{r=1}^{n}\left(\tan\left(\frac{x}{2^{r-1}}\right) - \tan\left(\frac{x}{2^r}\right)\right)$$

$$= \tan x - \tan\frac{x}{2} + \tan\frac{x}{2} - \tan\frac{x}{4} + \ldots - \tan\left(\frac{x}{2^n}\right)$$

$$f(x) = \tan x - \tan\left(\frac{x}{2^n}\right)$$

Now $$g(x) = \lim_{n\to\infty} \frac{\ell n \tan x - (\tan x)^n[\sin(\tan\frac{x}{2})]}{1+(\tan x)^n}$$

$$g(x) = \begin{cases} \ell n(\tan x) & \text{when } x < \frac{\pi}{4} \\ -\left[\sin(\tan\frac{x}{2})\right] & \text{when } x > \frac{\pi}{4}\end{cases}$$

$$g\left(\frac{\pi}{4}-h\right) = \lim_{h\to 0} \ell n\left(\tan\left(\frac{\pi}{4}-h\right)\right) = \ell n\ 1 = 0$$

$\Rightarrow K = 0$ and $g(x)$ is continuous in $(0, \frac{\pi}{2})$

5. $$g(0^-) = \lim_{h\to 0}\frac{1-a^{-h}+(-h)a^{-h}\ell n(a)}{a^{-h}(-h)^2} = \lim_{h\to 0}\frac{a^h - 1 - h\ \ell na}{h^2}$$

$$= \lim_{h\to 0}\frac{1+h\ell na + \frac{h^2}{2!}(\ell na)^2 + \ldots - 1 - h\ell na}{h^2} = \frac{(\ell na)^2}{2}$$

$$g(0^+) = \lim_{h\to 0}\frac{2^h a^h - h\ell n2 - h\ell na - 1}{h^2}$$

$$= \lim_{h\to 0}\frac{1+h\ell n(2a)+\frac{h^2}{2!}(\ell n2a)^2+\ldots-h\ell na - 1}{h^2} = \frac{(\ell n2a)^2}{2!}$$

Now $g(x)$ is continuous so

$(\ell na)^2 = (\ell n2a)^2$

$\Rightarrow \quad (\ell na)^2 = (\ell n\ 2)^2 + (\ell n\ a)^2 + 2\ell n\ 2\ell n\ a$

$\Rightarrow \quad \ell n\ a = \frac{-1}{2}\ell n\ 2 \qquad \Rightarrow \qquad a = \frac{1}{\sqrt{2}}$

$$g(0) = \frac{\left(\log\left(\frac{1}{\sqrt{2}}\right)\right)^2}{2} = \frac{1}{8}(\ell n\ 2)^2$$

6. $$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{(\frac{\pi}{2}-\sin^{-1}(1-\{h\}^2)).\sin^{-1}(1-\{h\})}{\sqrt{2}(\{h\}-\{h\}^3)}$$

$$= \lim_{h\to 0}\frac{(\frac{\pi}{2}-\sin^{-1}(1-h^2))\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-h^2)}{\sqrt{2}(1-h^2)}\times\frac{\sin^{-1}(1-h)}{h} = \frac{\pi}{2}$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{(\frac{\pi}{2}-\sin^{-1}(1-(1-h)^2)\sin^{-1}(1-(1-h))}{\sqrt{2}((1-h)-(1-h)^3)}$$

$$= \lim_{h\to 0}\frac{\frac{\pi}{2}\sin^{-1}h}{\sqrt{2}(1-h)(2-h)h} = \frac{\pi}{4\sqrt{2}}$$

so $f(x)$ is discontinuous at $x = 0$

Now $$g(x) = \begin{cases}\frac{\pi}{2} & ; \quad x \ge 0 \\ 2\sqrt{2}\,\frac{\pi}{4\sqrt{2}} & ; \quad x < 0\end{cases}$$

$$g(x) = \begin{cases}\frac{\pi}{2} & ; \quad x \ge 0 \\ \frac{\pi}{2} & ; \quad x < 0\end{cases}$$

so $g(x)$ is continuous at $x = 0$

9. Since g is onto continuous function so by reference of intermediate value theorem we get required result.

10. $$y_n(x) = x^2\,\frac{\left(\frac{1}{(1+x^2)^n}-1\right)}{\frac{1}{1+x^2}-1}$$

$$= (1+x^2)\left(1-\frac{1}{(1+x^2)^n}\right) \quad \text{when } x \ne 0,\ n \in N$$

$= 0 \qquad$ when $x = 0,\ n \in N$

$$y(x) = \lim_{n\to\infty} y_n(x) = \begin{cases}1+x^2 & x \ne 0 \\ 0 & x = 0\end{cases}$$

so $y(x)$ is discontinuous at $x = 0$

2. $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\therefore$ $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

so $f(x) = \begin{cases} xe^{-2/x}, & x > 0 \\ x, & x < 0 \\ 0, & x = 0 \end{cases}$

(I) continuous at x = 0

$\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(0-h) = f(0)$

$\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} h \quad e^{-2/h} = 0$

$\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} = 0 \qquad f(0) = 0$

f(x) is continuous at x = 0 or f(x) is continuous for all x

(II) differentiability at x = 0

L.H.D. = $Lf'(0) = \lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = \dfrac{-h-0}{-h} = 1$

R.H.D. $Rf'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$\Rightarrow \dfrac{h \times e^{-2/h} - 0}{h} = e^{-2/h} = 0$

$Lf'(0) \neq Rf'(0$

f(x) is not differentiable at x = 0

so that f(x) is cont at x = 0 but not differentiable at x = 0

3. $f(x) = \dfrac{1-\tan x}{4x-\pi} \quad x \neq \pi/4 \quad x \in [0, \pi/2]$

f(x) is continuous at $x \in [0, \pi/2]$

so at $x = \pi/4$

$\lim\limits_{h\to 0} f\left(\dfrac{\pi}{4}+h\right) = \lim\limits_{h\to 0} f\left(\dfrac{\pi}{4}-h\right) = f\left(\dfrac{\pi}{4}\right)$

so $\lim\limits_{h\to 0} f\left(\dfrac{\pi}{4}+h\right) = f\left(\dfrac{\pi}{4}\right)$

$\lim\limits_{h\to 0} f\left(\dfrac{\pi}{4}+h\right) = \dfrac{1-\tan x}{4x-\pi}$

$= \lim\limits_{h\to 0} \dfrac{1-\tan\left(\dfrac{\pi}{4}+h\right)}{4\left(\dfrac{\pi}{4}+h\right)-\pi}$

$= \lim\limits_{h\to 0} \dfrac{1-\left(\dfrac{\tan\frac{\pi}{4}+\tanh}{1-\tan\frac{\pi}{4}\tanh}\right)}{\pi+4h-\pi}$

$= \lim\limits_{h\to 0} \dfrac{1-\tanh-1-\tanh}{(1-\tanh)\times 4h}$

$= \lim\limits_{h\to 0} \dfrac{-2}{4}\left(\dfrac{\tanh}{h}\right) = -\dfrac{1}{2}$

$\lim\limits_{h\to 0} f\left(\dfrac{\pi}{4}+h\right) = -\dfrac{1}{2} = f\left(\dfrac{\pi}{4}\right)$

$f\left(\dfrac{\pi}{4}\right) = -\dfrac{1}{2}$

4. $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ can be continuous at x = 0

so $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(0-h) = f(0)$

$\lim\limits_{h\to 0} f(0+h) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$

$\lim\limits_{h\to 0} \dfrac{1}{h} - \dfrac{2}{e^{2h}-1}$

$\lim\limits_{h\to 0} \dfrac{(e^{2h}-1)-2h}{h\times(e^{2h}-1)} = \dfrac{0}{0}$ form

$\lim\limits_{h\to 0} \dfrac{e^{2h}\times 2-0-2}{h\times e^{2h}\times 2+e^{2h}-1} = \dfrac{0}{0}$ form

$\lim\limits_{h\to 0} \dfrac{2\times 2e^{2h}}{2e^{2h}+h\times e^{2h}\times 2\times 2+e^{2h}\times 2} = \dfrac{4}{4} = 1$

f(0) = 1

5. LHL $= \lim\limits_{x\to 0} \dfrac{\sin(p+1)x}{x} + \dfrac{\sin x}{x}$

= (p + 1) + 1 = p + 2

LHL = f(0) $\Rightarrow$ $\boxed{p+2=q}$ ...(1)

RHL $= \lim\limits_{x\to 0} \dfrac{x^2}{x^{3/2}(\sqrt{x+x^2}+\sqrt{x})} = \dfrac{1}{2}$

$p + 2 = q = \dfrac{1}{2} \Rightarrow q = \dfrac{1}{2}, \; p = \dfrac{-3}{2}$

6. $f_1(x) = x$; $x \in R$ is continuous.

$f_2(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

$\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist

$\therefore$ $f_2(x)$ is discontinuous on R.

Now, $f(x) = \begin{cases} f_1(x).f_2(x) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

$\Rightarrow \quad \lim\limits_{x\to 0} f_1(x).f_2(x) = \lim\limits_{x\to 0} x.\sin\left(\frac{1}{x}\right)$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 0 = f(0)$$

$\therefore$ $f(x)$ is continuous on R

$\therefore$ Statement–1 is true, statement–2 is false.

7. $f(x) = |x - 2| + |x - 5|$; $x \in R$

f(x) is continuous in [2, 5] and differentiable is (2, 5) and f(2) = f(5) = 3.

$\therefore$By Rolle's theorem f'(x) = 0 for at least one $x \in (2,5)$.

$$f'(x) = \frac{|x-2|}{x-2} + \frac{|x-5|}{x-5}$$

f'(4) = 0 but f'(x) = 0 $\forall$ $x \in (2, 5)$

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

2. For f to be continuous :

$f(2n^-) = f(2n^+)$.

$\Rightarrow \quad b_n + \cos 2n\pi = a_n + \sin 2n\pi$

$\Rightarrow \quad b_n + 1 = a_n$

$\Rightarrow \quad a_n - b_n = 1$

($\therefore$ B is correct)

Also $f(x) = \begin{bmatrix} b_n + \cos \pi x & (2n-1, 2n) \\ a_n + \sin \pi x & [2n, 2n+1] \\ b_{n+1} + \cos \pi x & (2n+1, 2n+2) \\ a_n + \sin \pi x & [2n+2, 2n+3] \end{bmatrix}$

Again $f((2n+1)^-) = f((2n+1)^+)$

$\Rightarrow \quad a_n = b_{n+1} - 1$

$\Rightarrow \quad a_n - b_{n+1} = -1$

$\Rightarrow \quad a_{n-1} - b_n = -1$ ($\therefore$ D is correct)