EXERCISE - 01 CHECK YOUR GRASP

2. $S = i + 2i^2 + 3i^3 + ... + 100\ i^{100}$

$iS = i^2 + 2i^3 + ... + 100\ i^{101}$

$S(1-i) = i + i^2 + i^3 + ... + i^{100} - 100\ i^{101}$

$$S = \frac{-100i}{1-i} = \frac{-100i(1+i)}{2} = -50(i-1) = 50(1-i)$$

7. $x^2 + (p + iq)\,x + 3i = 0$

$\alpha + \beta = -(p + iq),\ \alpha\beta = 3i$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = [-(p + iq)]^2 - 6i$

$= (p^2 - q^2) + i\,(2pq - 6) = 8$

$\Rightarrow p^2 - q^2 = 8$ and $pq = 3$

$\Rightarrow p = 3,\ q = 1$ or $p = -3,\ q = -1$

12. $|z_1| = 1,\ |z_2| = 2,\ |z_3| = 3$

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \left||z_3|^2 z_1z_2 + |z_2|^2 z_1z_3 + |z_1|^2 z_2z_3\right| = 12$

$\Rightarrow |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1z_2z_3\bar{z}_1| = 12$

$\Rightarrow |z_1z_2z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}|= 12$

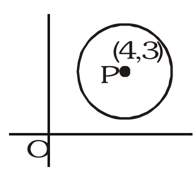
$\Rightarrow |z_1|\,|z_2|\,|z_3|\,|\overline{z_1 + z_2 + z_3}| = 12$

$\Rightarrow |z_1 + z_2 + z_3| = 2$

13. $|z - 4 - 3i| = 2$ represents a circle with centre (4, 3) and radius 2.

so minimum and maximum distances from origin will be

OP – r and OP + r respectively.



$|z|_{min} = 5 - 2 = 3$

$|z|_{max} = 5 + 2 = 7$

14. $z^2 + z + 1$ is real so

$z^2 + z + 1 = \bar{z}^2 + \bar{z} + 1$

$z^2 - \bar{z}^2 + z - \bar{z} = 0$

$(z - \bar{z})\,(z + \bar{z} + 1) = 0$

either $z = \bar{z}$ or $z + \bar{z} + 1 = 0$

$\Rightarrow$ Im(z) = 0 Let $z = \alpha + i\beta$

$\Rightarrow$ z is purely real so $\alpha + i\beta + \alpha - i\beta + 1 = 0$

$\Rightarrow 2\alpha + 1 = 0$

$\Rightarrow \alpha = -\frac{1}{2}$

Also $(\alpha + i\beta)^2 + (\alpha + i\beta) + 1 > 0$

$\alpha^2 + \alpha + 1 - \beta^2 + i(2\,\alpha\,\beta + \beta) > 0$

if $\alpha = -1/2$ then

$\frac{1}{4} - \frac{1}{2} + 1 - \beta^2 > 0$

$\Rightarrow \beta^2 - \frac{3}{4} < 0 \Rightarrow -\frac{\sqrt{3}}{2} < \beta < \frac{\sqrt{3}}{2}$

16. If in a complex number a + ib, the ratio a : b is $1 : \sqrt{3}$ is then always try to convert that complex number in ω.

Here $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Therefore,

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$= 4 + 5\omega^{334} + 3\omega^{365}$

$= 4 + 5\omega + 3\omega^2$ $(\because \omega^3 = 1)$

$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$

$= 1 + 2\omega + 3\,(1 + \omega + \omega^2) = 1 + 2\omega + 3\quad 0$

$(\because 1 + \omega + \omega^2 = 0)$

$= 1 + (-1 + \sqrt{3}\,i) = \sqrt{3}\,i.$

19. $z^2 + pz + q = 0$

$z_1 + z_2 = -p,\qquad z_1z_2 = q$

If z_1, z_2, z_3 are the vertices of an equilateral triangle

then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

If z_3 is origin then $z_1^2 + z_2^2 = z_1z_2$

$\Rightarrow (z_1 + z_2)^2 = 3z_1z_2 \Rightarrow p^2 = 3q$

22. Let $z = r\,(\cos\theta + i\sin\theta) = r\,e^{i\theta}$

So $\omega = \frac{1}{r}[\cos(\theta - \pi/2) + i\sin(\theta - \pi/2)] = \frac{1}{r}e^{i(\theta - \pi/2)}$

So $\bar{z}\,\omega = r\,e^{-i\theta}\ \frac{1}{r}\ e^{i(\theta - \pi/2)} = e^{-i\pi/2} = -i$

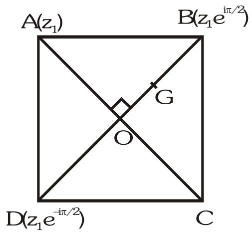
26. $1 - \log_2 \frac{|x+1+2i| - 2}{\sqrt{2} - 1} \ge 0$

$\Rightarrow \frac{|x+1+2i| - 2}{\sqrt{2} - 1} \le 2 \Rightarrow |x+1+2i| \le 2\sqrt{2}$

$\Rightarrow \sqrt{(x+1)^2 + 4} \le 2\sqrt{2} \Rightarrow (x+1)^2 + 4 \le 8$

$\Rightarrow (x+1)^2 \le 4 \Rightarrow -3 \le x \le 1$

But x = –1 not lie in the domain of function.

30. 

ΔABC is isosceles triangle

So centroid divide median BO in ratio 2 : 1

centroid $G = \frac{z_1 e^{i\pi/2}}{3} = \frac{z_1}{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Also centroid $G = \frac{z_1 e^{-i\pi/2}}{3} = \frac{z_1}{3}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

EXERCISE - 02 — BRAIN TEASERS

1. $\alpha = -2 + 3z$

$\alpha + 2 = 3z$

$|\alpha + 2| = 3|z|$

$(x + 2)^2 + y^2 = 9$

Similarly $\beta = -2 - 3z$

$\Rightarrow \beta + 2 = -3z$

$\Rightarrow |\beta + 2| = |-3z|$

$(x + 2)^2 + y^2 = 9$

Now $\alpha - \beta = 6z \Rightarrow |\alpha - \beta| = 6|z|$

so $(\alpha - \beta)$ moves on a circle with centre as origin and radius 6.

3. $z^3 + (1 + i)z^2 + (1 + i) z + i = 0$

$\Rightarrow (z + i)(z^2 + z + 1) = 0$

$\Rightarrow (z + i)(z - \omega)(z - \omega^2) = 0$

$\Rightarrow z = -i, \omega, \omega^2$

Now ω, and ω^2 satisfies the equation

$z^{1993} + z^{1994} + 1 = 0$

So ω and ω^2 are common roots

4. $\prod_{r=1}^{n} x_r = e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2^2}} \cdot e^{i\frac{\pi}{2^3}} \ldots\ldots e^{i\frac{\pi}{2^n}}$

$\lim_{n\to\infty} \prod_{r=1}^{n} x_r = e^{i\frac{\pi}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\right)}$

$\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = \text{Re}\left(e^{i\left(\frac{\pi}{2}\right)2}\right) = -1$

$\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = \text{Im}\left(e^{i\left(\frac{\pi}{2}\right)2}\right) = 0$

5. Since z_1 and z_2 lie on $|z| = 1$ and $|z| = 2$

then $|z_1| = 1$ and $|z_2| = 2$

$|2z_1 + z_2| \le 2|z_1| + |z_2| \le 4$

max $|2z_1 + z_2| = 4$

$|z_1 - z_2| \ge ||z_1| - |z_2|| = |1-2| = 1$

min $|z_1 - z_2| = 1$

$\left|z_2 + \frac{1}{z_1}\right| \le |z_2| + \frac{1}{|z_1|} = 2 + 1 = 3$

$\left|z_2 + \frac{1}{z_1}\right| \le 3$

9. $z = |z + i\omega| \le |z| + |\omega|$

$(\because |z_1 + z_2| \le |z_1| + |z_2|)$

$\therefore |z| + |\omega| \ge 2$ ... (i)

But given that $|z| \le 1$ and $|\omega| \le 1$

$\Rightarrow |z| + |\omega| \le 2$ ... (ii)

from (i) and (ii) $|z| = |\omega| = 1$

Also $|z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow |z + i\omega|^2 = |z - i\bar{\omega}|^2$

$\Rightarrow (z + i\omega)(\overline{z + i\omega}) = (z - i\bar{\omega})(\overline{z - i\bar{\omega}})$

$\Rightarrow (z + i\omega)(\bar{z} - i\bar{\omega}) = (z - i\bar{\omega})(\bar{z} + i\omega)$

$\Rightarrow z\bar{z} + i\omega\bar{z} - i\bar{\omega}z + \omega\bar{\omega} = z\bar{z} - i\bar{\omega}\bar{z} + i\omega z + \omega\bar{\omega}$

$\Rightarrow \omega\bar{z} - z\bar{\omega} + \bar{z}\bar{\omega} - \omega z = 0$

$\Rightarrow (\bar{z} - z)(\omega + \bar{\omega}) = 0$

$\Rightarrow z = \bar{z}$ or $\omega = -\bar{\omega}$

$\Rightarrow \text{Im}(z) = 0$ or $\text{Re}(\omega) = 0$

Also $|z| = 1$, $|\omega| = 1$

$\Rightarrow z = 1$ or -1 and $\omega = i$ or $-i$

EXERCISE - 03 — MISCELLANEOUS TYPE QUESTIONS

Match the column :

1. (A) $\left||z| - \frac{1}{|z|}\right| \le \left|z + \frac{1}{z}\right|$

$-2 \le |z| - \frac{1}{|z|} \le 2$

$|z|^2 + 2|z| - 1 \ge 0$ and $|z|^2 - 2|z| - 1 \le 0$

$|z| \ge \sqrt{2} - 1$, $|z| \le \sqrt{2} + 1$

$|z|_{min} = \sqrt{2} - 1$

so minimum value of $\frac{|z|}{\tan\frac{\pi}{8}} = 1$

(B) $|z| = 1$

Let $z = \cos\theta + i\sin\theta$

$$\frac{z^n}{z^{2n}+1} - \frac{\overline{z}^n}{\overline{z}^{2n}+1}$$

$$= \frac{\cos n\theta + i\sin n\theta}{1+\cos 2n\theta + i\sin 2n\theta} - \frac{\cos n\theta - i\sin n\theta}{1+\cos 2n\theta - i\sin 2n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta}{2\cos n\theta(\cos n\theta + i\sin n\theta)}$$

$$- \frac{\cos n\theta - i\sin n\theta}{2\cos n\theta(\cos n\theta - i\sin n\theta)}$$

$$= \frac{1}{2\cos n\theta} - \frac{1}{2\cos n\theta} = 0$$

(C) $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow$ $(2iz + 3)(4z^2 + 9i) = 0$

$\Rightarrow$ $z = \frac{3}{2}i,\ z^2 = -\frac{9}{4}i$ $\Rightarrow$ $2|z| = 3$

(D) $z^4 + z^3 + z^2 + z + 1$

$= (z - z_1)(z - z_2)(z - z_3)(z - z_4)$

Put $z = -2$

$$\prod_{i=1}^{4}(z_i + 2) = (-2)^4+(-2)^3+(-2)^2+(-2)+1 = 11$$

Assertion & Reason :

1. $|z - 4 - 5i| = 4$ represents a circle with centre $(4, 5)$ and radius 4 and $\arg(z - 3 - 4i) = \frac{\pi}{4}$ represents a ray emanating from point (3, 4). Ray will intersect the circle at only one point.

So statement (I) is false and statement (II) is true.

Comprehension # 2

1. $AD = x$, $\angle ADC = 180 - (C + \theta)$

Area of $\Delta ABC = 2$ area $\Delta ADC = \frac{1}{2}2y.x\sin(C+\theta)$

$= xy\sin(C + \theta)$

2. Let affix of M is z_m and $\angle BOM = \pi - 2B$, then

$$\frac{z_m - 0}{z_b - 0} = \frac{OM}{OB}e^{i(\pi-2B)}$$

$z_m = z_b\ e^{i(\pi - 2B)}$

3. Let affix of L is z_L and $\angle BOL = 2(A - \theta)$, then

$$\frac{z_L - 0}{z_b - 0} = e^{i(2A-2\theta)}$$

$z_L = z_b\ e^{i(2A-2\theta)}$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

3. $|z_1 - 2z_2| = |2 - z_1\overline{z}_2|$

$|z_1 - 2z_2|^2 = |2 - z_1\overline{z}_2|^2$

$(z_1 - 2z_2)\left(\overline{z}_1 - 2\overline{z}_2\right) = \left(2 - z_1\overline{z}_2\right)\left(2 - \overline{z}_1 z_2\right)$

$z_1\overline{z} - 2\overline{z}_1 z_2 - 2z_1\overline{z}_2 + 4z_2\overline{z}_2$

$= 4 - 2z_1\overline{z}_2 - 2\overline{z}_1 z_2 + z_1\overline{z}_1 z_2\overline{z}_2$

$|z_1|^2 + 4|z_2|^2 - 4 - |z_1|^2|z_2|^2 = 0$

$\Rightarrow$ $(|z_1|^2 - 4)(1 - |z_2|^2) = 0$

$\Rightarrow$ $|z_1| = 2$ (as $|z_2| \neq 1$)

4. $iz^3 + z^2 - z + i = 0$

$\Rightarrow$ $z^3 - iz^2 + iz + 1 = 0$ $\Rightarrow$ $(z - i)(z^2 + i) = 0$

$\Rightarrow$ $z^2 = -i$ or $z = i$ $\Rightarrow$ $|z| = 1$

5. $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$

$= e^{-iA}\ e^{-iB}\ e^{-iC}\begin{vmatrix} e^{-iA} & e^{i(A+C)} & e^{i(A+B)} \\ e^{i(B+C)} & e^{-iB} & e^{i(A+B)} \\ e^{i(B+C)} & e^{i(A+C)} & e^{-iC} \end{vmatrix}$

As $A + B + C = \pi$

So $A + C = \pi - B$, $B + C = \pi - A$, $A + B = \pi - C$

$D = e^{-i\pi}\begin{vmatrix} e^{-iA} & e^{i(\pi-B)} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{-iB} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{i(\pi-B)} & e^{-iC} \end{vmatrix}$

$D = -\begin{vmatrix} e^{-iA} & -e^{iB} & -e^{-iC} \\ -e^{iA} & e^{-iB} & -e^{iC} \\ -e^{-iA} & -e^{-iB} & e^{-iC} \end{vmatrix}$

$= -e^{-iA}\ e^{-iB}\ e^{-iC}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -e^{-i\pi}(-4) = -4$

6. $|z|^2\omega - |\omega|^2 z = z - \omega$ .....(i)

Put $z = \omega$ & $z = \frac{1}{\overline{\omega}}$

we get L.H.S = R.H.S

Now, equation (i) be written as

$\omega(1 + |z|^2) = z(1 + |\omega|^2)$

$\Rightarrow$ $\frac{\omega}{z} = \frac{(1+|\omega|^2)}{1+|z|^2} = \lambda$ $\Rightarrow$ $\omega = \lambda z$

But this is equation (i)

$|z|^2\lambda z - \lambda^2|z|^2 z = z - \lambda z$

$\Rightarrow$ $z\lambda|z|^2(1 - \lambda) = z(1 - \lambda)$

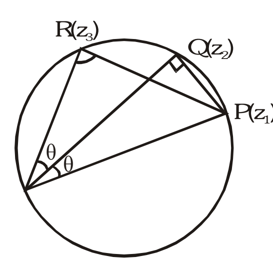
$\Rightarrow$ $(1 - \lambda)(\lambda|z|^2 - 1) = 0$

$\Rightarrow$ $\lambda = 1$; $\lambda = \frac{1}{|z|^2}$

from $\lambda = 1$ we get $z = \omega$

$\lambda = \frac{1}{|z|^2}$ we get $\omega = \frac{1}{\overline{z}}$

11. From the fig.



we have

$z_2 = z_1(\cos\theta\ e^{i\theta})$

and $z_3 = z_1(\cos 2\theta\ e^{i2\theta})$

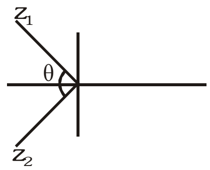
$\Rightarrow$ $\frac{z_2^2}{z_3} = \frac{(z_1\cos\theta)^2}{(z_1\cos 2\theta)}$

$\Rightarrow$ $z_2^2\cos 2\theta = z_1 z_3\cos^2\theta$

4. $\frac{z_1}{z_2} = e^{i\theta}$

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{\cos\theta + 1 + i\sin\theta}{\cos\theta - 1 + i\sin\theta}$$

$$= \frac{2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}}{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$



$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{\cot(\theta/2)}{i}$$

$$i\tan\frac{\theta}{2} = \left(\frac{z_1 - z_2}{z_1 + z_2}\right)$$

$$-\tan^2\frac{\theta}{2} = \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2 \Rightarrow 1 - \sec^2\frac{\theta}{2} = \frac{\frac{b^2}{a^2} - \frac{4c}{a}}{\frac{b^2}{a^2}}$$

$$\Rightarrow \quad 1 - \sec^2\frac{\theta}{2} = 1 - \frac{4ac}{b^2} \quad \Rightarrow \quad \cos^2\frac{\theta}{2} = \frac{b^2}{4ac}$$

$$\Rightarrow \quad \cos\frac{\theta}{2} = \sqrt{\frac{b^2}{4ac}} \quad \Rightarrow \quad \theta = 2\cos^{-1}\sqrt{\frac{b^2}{4ac}}$$

5. Let $z^{2m} + z^{2m-1} - 1 ... + z + 1 = (z - z_1)(z - z_2).....(z - z_{2m})$

Taking log on both the sides & differentiating w.r.t.z

$$\frac{2mz^{2m-1} + (2m-1)z^{2m-2} + ... + 2z + 1}{z^{2m} + z^{2m-1} + ... + z^2 + z + 1}$$

$$= \frac{1}{z - z_1} + \frac{1}{z - z_2} + ... + \frac{1}{z - z_{2m}}$$

$$\Rightarrow \frac{1 + 2 + 3 + ... + 2m}{(2m+1)} \qquad \text{(put } z = 1)$$

$$= \frac{1}{1 - z_1} + \frac{1}{1 - z_2} + ... + \frac{1}{1 - z_{2m}}$$

$$\Rightarrow \sum_{r=1}^{2m} \frac{1}{z_r - 1} = -\left[\frac{2m(2m+1)}{2(2m+1)}\right] = -m$$

6. (a) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ ... (1)

Put $x = i$

$(1 + i)^n = C_0 + C_1i - C_2 - C_3i + C_4 + ... + C_ni^n$

$$\Rightarrow 2^{n/2}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]$$

$= (C_0 - C_2 + C_4 - C_6 + ...) + i(C_1 - C_3 + C_5 ...)$

$$\Rightarrow C_0 - C_2 + C_4 + ... = 2^{n/2}\cos\frac{n\pi}{4} \quad ... (2)$$

$$\Rightarrow C_1 - C_3 + C_5 + ... = 2^{n/2}\sin\frac{n\pi}{4} \quad ... (3)$$

Again put $x = 1$ and -1 in the equation (1)

$2^n = C_0 + C_1 + C_2 + C_3 + ... + C_n$

$0 = C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n$

Adding

$2^n = 2(C_0 + C_2 + C_4 + ...)$

$C_0 + C_2 + C_4 + C_6 + ... = 2^{n-1}$ ... (4)

Adding (2) and (4)

$$C_0 + C_4 + C_8 + ... + = \frac{1}{2}\left[2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}\right]$$

(e) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Put $x = 1, \omega, \omega^2$

$2^n = C_0 + C_1 + C_2 + ... + C_n$

$(1 + \omega)^n = C_0 + C_1\omega + C_2\omega^2 + ... + C_n\omega^n$

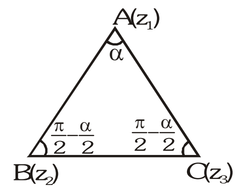
$(1 + \omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + ... + C_n\omega^{2n}$

Adding

$3(C_0 + C_3 + C_6 + ... +) = 2^n + (-\omega^2)^n + (-\omega)^n$

$$C_0 + C_3 + C_6 + ... = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

8. $\frac{z_3 - z_1}{z_2 - z_1} = e^{i\alpha}$



$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = \cos\alpha - 1 + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = 2i\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

squaring both sides

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)$$

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

$\Rightarrow (z_3 - z_2)^2 = 4\sin^2(\alpha/2)(z_3 - z_1)(z_1 - z_2)$

9. $$\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11} = -i\left[\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right]$$

$$= -ie^{i\frac{2q\pi}{11}}$$

$$\therefore \sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right) = -i\sum_{k=1}^{10} e^{i\frac{2q\pi}{11}}$$

$$= -i\left[e^{\frac{2\pi i}{11}} + e^{\frac{4\pi i}{11}} + ... + e^{\frac{20\pi i}{11}}\right] = -i(-1) = i$$

Given expression

$$\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p$$

$$= \sum_{p=1}^{32} (3p+2)i^p = 3\sum_{p=1}^{32} pi^p + 2\sum_{p=1}^{32} i^p = 3S_1 + 2S_2$$

$$\text{where } S_1 = \sum_{p=1}^{32} pi^p$$

$$S_1 = i + 2i^2 + 3i^3 + \ldots + 32i^{32}$$

$$iS_1 = i^2 + 2i^3 + \ldots + 32i^{33}$$

$$S_1(1-i) = i + i^2 + i^3 + \ldots + i^{32} - 32i^{33}$$

$$S_1 = \frac{-32i}{(1-i)} = 16(1-i)$$

$$S_2 = 0$$

$$\therefore \sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p$$

$$= 3S_1 + 2S_2 = 48(1-i) + 0 \qquad = 48(1-i)$$

EXERCISE - 05 [A] JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

6. Given thar arg $z\omega = \pi$ ... (i)

$\overline{z} + i\overline{\omega} = 0 \Rightarrow \overline{z} = -i\overline{\omega} \Rightarrow z = i\omega \Rightarrow \omega = -iz$

From (i) arg $(-iz)^2 = \pi$

arg $(-i)$ + 2 arg $(z) = \pi$; $\frac{-\pi}{2}$ + 2 arg $(z) = \pi$

2 arg $(z) = \frac{3\pi}{2}$; arg $(z) = \frac{3\pi}{4}$

10. Given that $\omega = \dfrac{z}{z - \frac{i}{3}}$ and $|\omega| = 1$

$$\therefore \quad |\omega| = \left|\frac{z}{z-\frac{i}{3}}\right| \Rightarrow \frac{|z|}{\left|z-\frac{i}{3}\right|} = 1$$

$$\Rightarrow |z| = \left|z - \frac{i}{3}\right| \Rightarrow -\frac{2}{3}y + \frac{1}{9} = 0$$

Which is a straight line.

13. $\left|Z - \frac{4}{Z}\right| \geq |Z| - \left|\frac{4}{Z}\right|$

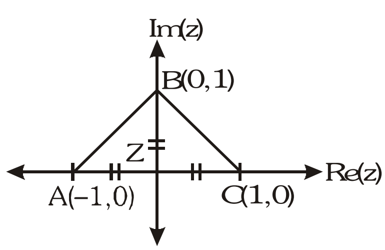
$$2 \geq |Z| - \frac{4}{|Z|}$$

$$2|Z| \geq |Z|^2 - 4$$

$$|Z|^2 - 2|Z| - 4 \leq 0$$

$$|Z| \leq \sqrt{5} + 1$$

14. z is the circumcentre (0, 0) of triangle ABC so their exist only one complex number.



15. Let $z^2 + \alpha z + \beta = 0$ has $(1 + iy_1)$ and $(1 + iy_2)$

so $z_1z_2 = \beta$

$(1 + iy_1)(1 + iy_2) = \beta$

$\beta = 1 - y_1y_2 + i(y_1 + y_2)$ ($\because$ β is purely real)

here $y_1 + y_2 = 0$

$y_1 = -y_2$

$\beta = 1 - y_1y_2$

$\beta = 1 + y_1^2$

$\beta > 1$

$\Rightarrow \beta \in (1, \infty)$

16. $(1 + \omega)^7 = A + B\omega$

$(-\omega^2)^7 = A + B\omega$

$-\omega^2 = A + B\omega$

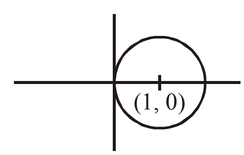
$1 + \omega = A + B\omega$

$A = 1$

$B = 1 \qquad (1, 1)$

17 $\dfrac{z^2}{z-1}$ is purely real where $(Z \neq 1)$

so $\dfrac{\overline{z}^2}{\overline{z}-1} = \dfrac{z^2}{z-1}$



$$z\overline{z}^2 - \overline{z}^2 = \overline{z}z^2 - z^2$$

$$z\overline{z}(z - \overline{z}) = z^2 - \overline{z}^2$$

$$z\overline{z}(z - \overline{z}) = (z + \overline{z})(z - \overline{z})$$

$$\Rightarrow \overline{z} - z = 0 \text{ or } z + \overline{z} = z\overline{z}$$

$$\Rightarrow \overline{z} = z \text{ or } x^2 + y^2 - 2x = 0$$

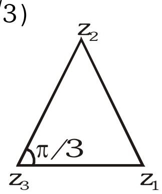
$$(x-1)^2 + y^2 = 1$$

so either lie on z real axis or on a circle passing through the origin.

18 $\overline{z} = \dfrac{1}{z}$

$$\arg\left(\frac{1+z}{1+\frac{1}{z}}\right) \Rightarrow \arg z \Rightarrow \theta$$

2. **(a)** $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$



$= \dfrac{1 - i^2.3}{2(1 + i\sqrt{3})}$

$= \dfrac{4}{2(1 + i\sqrt{3})} = \dfrac{2}{(1 + i\sqrt{3})}$

$\Rightarrow \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\Rightarrow \left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$ and $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$

Hence the Δ is equilateral,

(b) $\arg\dfrac{z_1}{z_2} = \dfrac{\pi}{2} \Rightarrow \dfrac{z_1}{z_2} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$

($\because$ $|z_2| = |z_1| = 1$)

$\therefore \dfrac{z_1^n}{z_2^n} = (i)^n$

Hence $i^n = 1 \Rightarrow n = 4k$

3. **(c)** $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow (z^p - 1)(z^q - 1) = 0$

as α is root of (1), either $\alpha^p - 1 = 0$

or $\alpha^q - 1 = 0$

$\Rightarrow$ either $\dfrac{\alpha^p - 1}{\alpha - 1} = 0$ or $\dfrac{\alpha^q - 1}{\alpha - 1} = 0$ (as $\alpha \neq 1$)

$\Rightarrow$ either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$

or $1 + \alpha + \ldots + \alpha^{q-1} = 0$

But $\alpha^p - 1 = 0$ and $\alpha^q - 1 = 0$

cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1 = \alpha^p = \alpha^q$.

5. Given $|z_1| < 1$ and $|z_2| > 1$

Then to prove $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$ $\left\{\text{using } \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right\}$

$\Rightarrow |1 - z_1\bar{z}_2| < |z_1 - z_2|$

Squaring both sides, we get

$(1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

{using $|z|^2 = z\bar{z}$ }

$\Rightarrow 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$\Rightarrow 1 + |z_1|^2\,|z_2|^2 < |z_1|^2 + |z_2|^2$

$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2\,|z_2|^2 < 0$

$\Rightarrow (1 - |z_1|^2)(1 - |z_2|^2) < 0$ ... (2)

which is true by (1) as $|z_1| < 1$ and $|z_2| > 1$

$\therefore$ $(1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

$\therefore$ (2) is true whenever (1) is true.

$\Rightarrow \left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$

6. Given : $a_1 z + a_2 z^2 + \ldots + a_n z^n = 1$

and $|z| < 1/3$ ... (1)

{using $|z_1 + z_2| \le |z_1| + |z_2|$}

$\Rightarrow |a_1 z| + |a_2 z^2| + |a_3 z^3| + \ldots + |a_n z^n| \ge 1$

$\Rightarrow 2\{(|z| + |z|^2 + |z|^3 + \ldots + |z|^n)\} > 1$

(using $|a_r| < 2$)

$\Rightarrow \dfrac{2|z|(1 - |z|^n)}{1 - |z|} > 1$

{using sum of n terms of G.P.}

$\Rightarrow 2|z| - 2|z|^{n+1} > 1 - |z|$

$\Rightarrow 3|z| > 1 + 2|z|^{n+1} \Rightarrow |z| > \dfrac{1}{3} + \dfrac{2}{3}|z|^{n+1}$

$\Rightarrow |z| > \dfrac{1}{3}$, which contradicts ... (1)

$\therefore$ There exists no complex number z such that

$|z| < \dfrac{1}{3}$ and $\sum\limits_{r=1}^{n} a_r z^r = 1$

8. As we know; $|z|^2 = z.\bar{z}$ $\Rightarrow \dfrac{|z - \alpha|^2}{|z - \beta|^2} = k^2$

$\Rightarrow (z - a)(\bar{z} - \bar{\alpha}) = k^2 (z - \beta)(\bar{z} - \bar{\beta})$

$|z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2 = k^2(|z|^2 - \beta\bar{z} - \bar{\beta}z + |\beta|^2)$

or $|z|^2(1 - k^2) - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2)z$

$+ (|\alpha|^2 - k^2|\beta|^2) = 0$

$\Rightarrow |z|^2 - \dfrac{(\alpha - k^2\beta)}{(1 - k^2)}\bar{z} - \dfrac{(\bar{\alpha} - \bar{\beta}k^2)}{(1 - k^2)}z + \dfrac{|\alpha|^2 - k^2|\beta|^2}{(1 - k^2)} = 0$

On comparing with equation of circle.

$|z|^2 + a\bar{z} + \bar{a}z + b = 0$

whose centre is (–a) and radius $= \sqrt{|a|^2 - b}$

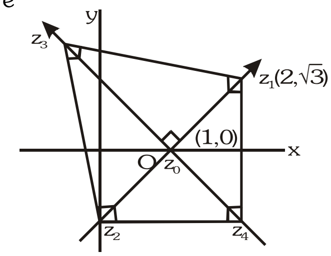
$\therefore$ centre for (i)

$= \dfrac{\alpha - k^2\beta}{1 - k^2}$ and radius

$= \sqrt{\left(\dfrac{\alpha - k^2\beta}{1 - k^2}\right)\left(\dfrac{\bar{\alpha} - k^2\bar{\beta}}{1 - k^2}\right) - \dfrac{\alpha\bar{\alpha} - k^2\beta\bar{\beta}}{1 - k^2}}$

radius $= \left|\dfrac{k(\alpha - \beta)}{1 - k^2}\right|$

13. Here, centre of circle is (1, 0) is also the mid-point of diagonals of square



$\Rightarrow \dfrac{z_1+z_2}{2} = z_0$

$\Rightarrow z_2 = -\sqrt{3}\,i$, (where $z_0 = 1 + 0\,i$)

and $\dfrac{z_3-1}{z_1-1} = e^{\pm i\pi/2}$

$\Rightarrow z_3 = 1+(1+\sqrt{3}i).\left(\cos\dfrac{\pi}{2} \pm i\sin\dfrac{\pi}{2}\right)$, as $z_1 = 2+\sqrt{3}i$

$= 1 \pm i(1+\sqrt{3}i) = (1 \mp \sqrt{3}) \pm i$

$z_3 = (1-\sqrt{3}) + i$ and $z_4 = (1+\sqrt{3}) - i$

12. Let, $z_1 = \dfrac{w-\bar{w}z}{1-z}$, be purely real

$\Rightarrow z_1 = \bar{z}_1$

$\therefore \dfrac{w-\bar{w}z}{1-z} = \dfrac{\bar{w}-w\bar{z}}{1-\bar{z}}$

$\Rightarrow w - w\bar{z} - \bar{w}z + \bar{w}z.\bar{z} = \bar{w} - z\bar{w} - w\bar{z} + wz.\bar{z}$

$\Rightarrow (w-\bar{w}) - (\bar{w}-w)|z|^2 = 0$

$\Rightarrow (w-\bar{w})(1-|z|^2) = 0$

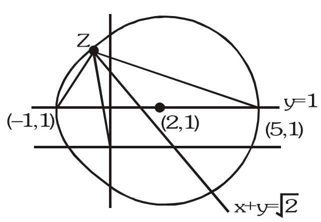
$\Rightarrow |z|^2 = 1$ {as, $w-\bar{w} \neq 0$, since $\beta \neq 0$}

$\Rightarrow |z| = 1$ and $z \neq 1$.

15. $A = \{z : \text{Im } z \geq 1\}$ $\quad y \geq 1$

$B = \{z : |z-2-i| = 3\}$ $\quad (x-2)^2 + (y-1)^2 = 9$

$C = \{z : \text{Re}((1-i)z\} = \sqrt{2}\}$ $\quad x + y = \sqrt{2}$



As we can see 3 curves intersects at only one point

So $A \cap B \cap C$ contains exactly one element

16. $|z+1-i|^2 + |z-5-i|^2 = (-1-5)^2 + (1-1)^2 = 36$

so exactly 36

17. As $3-\sqrt{5} \leq |z| \leq 3+\sqrt{5}$

As $-3+\sqrt{5} \leq |\omega| \leq 3+\sqrt{5}$

$-3-\sqrt{5} \leq -|\omega| \leq 3-\sqrt{5}$

$-\sqrt{5} \leq -|\omega| + 3 \leq 6-\sqrt{5}$

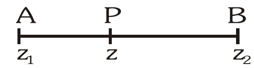
$-3 \leq |z| - |\omega| + 3 \leq 9$

22. $z = z_1 + t(z_2 - z_1)$

$\dfrac{z-z_1}{z_2-z_1} = t$, $t \in (0,1) \Rightarrow z = \dfrac{z_1(1-t)+tz_2}{(1-t)+t}$

point P(z) divides point $A(z_1)$ & $B(z_2)$ internally in ratio $(1-t) : t$

Hence locus is a line segment such that P(z) lies between $A(z_1)$ & $B(z_2)$ as shown in figure.



Hence options A,C & D are correct.

24. (A) $|z - i|z||^2 = |z + i|z||^2$

$\Rightarrow (z - i|z|)\left(\bar{z} + i|z|\right) = (z + i|z|)(\bar{z} - i|z|)$

$\Rightarrow 2i|z|z = 2i|z|\bar{z}$

$\Rightarrow z = \bar{z} \therefore z$ is purely real.

$\therefore$ z lies on real axis.

(B) Locus is ellipse having focii (–4, 0) & (4, 0)

$2ae = 8$ & $2a = 10$

$\Rightarrow a = 5$ & $e = 4/5$

It is ellipse having eccentricity 4/5.

(C) $w = 2(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\theta + i\sin\theta\right) - \dfrac{1}{2(\cos\theta + i\sin\theta)}$

$x + iy = \dfrac{3}{2}\cos\theta + \dfrac{i5}{2}\sin\theta$

$\Rightarrow x = \dfrac{3}{2}\cos\theta$ & $y = \dfrac{5}{2}\sin\theta$

It is a locus $\dfrac{x^2}{9/4} + \dfrac{y^2}{25/4} = 1$

$\dfrac{9}{4} = \dfrac{25}{4}(1-e^2) \Rightarrow e = \dfrac{4}{5}$

since $x = \dfrac{3}{2}\cos\theta \Rightarrow |\text{Re}(z)| \leq \dfrac{3}{2}$

$|\text{Re}(z)| \leq \dfrac{3}{2} \Rightarrow |\text{Re}(z)| \leq 2$

Consider the circle $x^2 + y^2 - 9 = 0$

By putting $x = \dfrac{3}{2}\cos\theta$

& $y = \dfrac{5}{2}\sin\theta$ into $x^2 + y^2 - 9$

$\dfrac{9\cos^2\theta}{4} + \dfrac{25}{4}\sin^2\theta - 9 < 0$

(D) $z = (\cos\theta + i\sin\theta) + \dfrac{1}{(\cos\theta + i\sin\theta)}$

$z = 2\cos\theta$

where z is real value & $z \in [-2, 2]$

25. **Comprehension (3 questions together)**

$a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$7a + 7b + 7c = 0$

$\Rightarrow a = K,\ b = 6K,\ c = -7K$

(i) (K, 6K, –7K)

$2x + y + z = 1$

$2K + 6K - 7K = 1$

($\because$ point lies on the plane)

$\Rightarrow K = 1$

$\Rightarrow 7a + b + c = 7K + 6K - 7K = 6$

(ii) $x^3 - 1 = 0$

$\Rightarrow x = 1, \omega, \omega^2$

$$\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad \text{since } \operatorname{Im}(\omega) > 0$$

If $a = 2 = K \Rightarrow b = 12$ & $c = -14$

Hence $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c} = \dfrac{3}{\omega^2} + \dfrac{1}{\omega^{12}} + \dfrac{3}{\omega^{-14}}$

$= 3\omega + 1 + 3\omega^2 = -3 + 1 = -2$

(iii) $\because$ $b = 6 \Rightarrow 6K = 6 \Rightarrow K = 1$

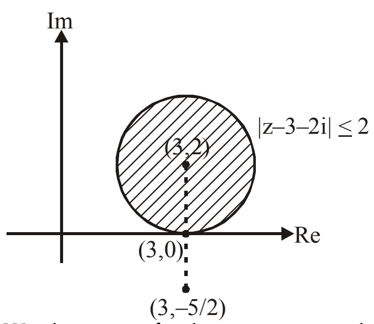
$\Rightarrow a = 1, \quad b = 6 \quad \& \quad c = -7$

$x^2 + 6x - 7 = 0$

$\Rightarrow \alpha + \beta = -6$, $\alpha\beta = -7$

$$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{\alpha+\beta}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1-\frac{6}{7}} = 7$$

26.



We have to find minimum value of

$$2\left|z - \left(3 - \frac{5}{2}i\right)\right|$$

$= 2$ (minimum distance between z and point $\left(3, -\frac{5}{2}\right)$)

$= 2$ (distance between (3,0) and $\left(3, -\frac{5}{2}\right)$)

$= 2 \quad \frac{5}{2} = 5$ units.

27. Ans. 3 (Bonus)

(Comment : If $\omega = e^{i\pi/3}$

then $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is not always an integer.

For example if a = b = c = 1 then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is $\dfrac{17}{3}$. Now if we consider $\omega = e^{i\,2\pi/3}$ then the solution is)

$|x|^2 = (a + b + c)\left(\bar{a} + \bar{b} + \bar{c}\right)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b} + a\bar{c} + b\bar{a} + b\bar{c} + c\bar{a} + c\bar{b}$

$|y|^2 = (a + b\omega + c\omega^2)\left(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega\right)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega^2 + a\bar{c}\omega + b\bar{a}\omega + b\bar{c}\omega^2 + c\bar{a}\omega^2 + c\bar{b}\omega$

$|z|^2 = (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$

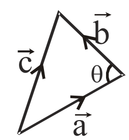
$= |a|^2 + |b|^2 + |c^2|$

$+ a\bar{b}\omega + a\bar{c}\omega^2 + b\bar{a}\omega^2 + b\bar{c}\omega + c\bar{a}\omega + c\bar{b}\omega^2$

$\therefore |x|^2 + |y|^2 + |z|^2 = 3(|a|^2 + |b|^2 + |c|^2)$

$$\Rightarrow \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$$

28. (A)



$|\vec{a}| = 2$

$|\vec{b}| = 2$

$|\vec{c}| = 2\sqrt{3}$

$$\cos\theta = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{a}||\vec{b}|}$$

$$= \frac{4+4-12}{2.2.2} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

(B) $\displaystyle\int_a^b (f(x) - 3x)dx = a^2 - b^2 = \int_a^b (-2x)dx$

$\Rightarrow \displaystyle\int_a^b (f(x) - x)dx = 0$

$\Rightarrow$ one of the possible solution of this equation is

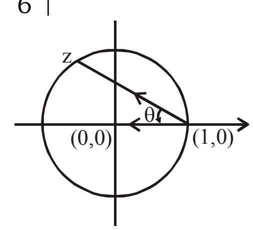
$f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

(C) $\displaystyle\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} (\sec \pi x)dx$

$$= \frac{\pi^2}{\ell n 3}\frac{1}{\pi}\left[\ell n|\sec \pi x + \tan \pi x|\right]_{7/6}^{5/6}$$

$$= \frac{\pi}{\ell n 3}\ell n\left|\frac{\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}}{\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}}\right| = \frac{\pi}{\ell n 3}\ell n\ 3 = \pi$$

(D) Let $\theta = \operatorname{Arg}\left(\dfrac{1}{1-z}\right)$



$\Rightarrow \theta = \operatorname{Arg}\left(\dfrac{0-1}{z-1}\right)$

which is shown in adjacent diagram.

$\Rightarrow$ Maximum value of θ is approaching to $\dfrac{\pi}{2}$ but θ will never obtained the value equal to $\dfrac{\pi}{2}$.

Hence there is an error in aksing the problem.

29. (A) Let $z = \cos\theta + i\sin\theta$

$$\text{Re}\left(\frac{2i(\cos\theta + i\sin\theta)}{1-(\cos\theta+i\sin\theta)^2}\right) = \text{Re}\left(\frac{\cos\theta i - \sin\theta}{\sin^2\theta - i\cos\theta\sin\theta}\right)$$

$$= \text{Re}\left(-\frac{1}{\sin\theta}\right) = \frac{-1}{\sin\theta}$$

$\therefore$ Set will be $(-\infty, -1] \cup [1, \infty)$

(B) $-1 \le \dfrac{8.3^{(x-2)}}{1-3^{2(x-1)}} \le 1 \qquad x \ne 1$

$\Rightarrow \quad -1 \le \dfrac{8.3^x}{(3-3^x)(3+3^x)} \le 1$

$3^x = t \qquad \therefore \quad t > 0$

$\dfrac{8t}{(3-t)(t+3)} \ge -1$

$\Rightarrow \quad t \in (0,3) \cup [9, \infty)$

$\Rightarrow \quad x \in (-\infty, 1) \cup [2, \infty)$

$\dfrac{8t}{(3-t)(t+3)} \le 1$

$\Rightarrow \quad t \in (0, 1] \cup (3, \infty)$

$\Rightarrow \quad x \in (-\infty, 0] \cup (1, \infty)$

Taking intersection,

$x \in (-\infty, 0] \cup [2, \infty)$

(C) $f(\theta) = \begin{vmatrix} 1 & \tan & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$C_1 \to C_1 + C_3$

$\Rightarrow \quad f(\theta) = \begin{vmatrix} 2 & \tan\theta & 1 \\ 0 & 1 & \tan\theta \\ 0 & -\tan\theta & 1 \end{vmatrix}$

$\Rightarrow \quad f(\theta) = 2\sec^2\theta$

$\Rightarrow \quad f(\theta) \in [2, \infty)$

(D) $f(x) = 3x^{5/2} - 10x^{3/2}$

$f'(x) = \dfrac{15}{2}x^{3/2} - \dfrac{30}{2}x^{1/2} > 0$

$\Rightarrow \quad \dfrac{15}{2}\sqrt{x}(x-2) \ge 0 \qquad \Rightarrow \quad x \ge 2$

30. $z^2 + z + 1 - a = 0$

$\because$ z is imaginary $\Rightarrow D < 0$

$1 - 4(1 - a) < 0$

$4a < 3$

$a < \dfrac{3}{4}$.

Aliter : $a = z^2 + z + 1$

$\because \quad a = \bar{a}$ (given a is real)

$\therefore \quad z^2 + z = \bar{z}^2 + \bar{z}$

$\Rightarrow \quad z^2 - \bar{z}^2 = \bar{z} - z$

$\Rightarrow \quad z + \bar{z} = -1$ ($\because$ Im(z) is non zero)

$\Rightarrow \quad \text{Re}(z) = -\dfrac{1}{2}$

$\therefore \quad$ z can be taken as $-\dfrac{1}{2} + iy$

where $y \in R$

$\therefore \quad a = \left(-\dfrac{1}{2} + iy\right)^2 + \left(\dfrac{-1}{2} + iy\right) + 1$

$\Rightarrow \quad a = \dfrac{1}{4} - \dfrac{1}{2} + 1 - iy + iy - y^2$

$\Rightarrow \quad a = \dfrac{3}{4} - y^2 \Rightarrow a < \dfrac{3}{4}$

$\therefore \quad a \ne \dfrac{3}{4}$

31. Given : α satisfies $|z - z_0| = r$

$\Rightarrow |\alpha - z_0| = r \qquad \ldots(1)$

& $\dfrac{1}{\alpha}$ satisfies $|z - z_0| = 2r$

$\Rightarrow \left|\dfrac{1}{\alpha} - z_0\right| = 2r \qquad \ldots(2)$

squaring (1) and (2) we get $(\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$\Rightarrow \alpha\bar{\alpha} - z_0\bar{\alpha} - \alpha\bar{z}_0 + z_0\bar{z}_0 = r^2 = 2|z_0|^2 - 2 \qquad \ldots(3)$

& $\left(\dfrac{1}{\bar{\alpha}} - z_0\right)\left(\dfrac{1}{\alpha} - \bar{z}_0\right) = 4r^2$

$\Rightarrow \dfrac{1}{\alpha\bar{\alpha}} - \dfrac{z_0}{\alpha} - \dfrac{\bar{z}_0}{\bar{\alpha}} + z_0\bar{z}_0 = 4r^2$

$\Rightarrow 1 - z_0\bar{\alpha} - \bar{z}_0\alpha + |z_0|^2|\alpha|^2 = 4\left(2|z_0|^2 - 2\right)|\alpha|^2$

$\Rightarrow 1 + 2|z_0|^2 - 2 - |\alpha|^2 - |z_0|^2 + |z_0|^2|\alpha|^2$

$= 8|z_0|^2|\alpha|^2 - 8|\alpha|^2$

$\Rightarrow -1 + |z_0|^2 - 7|z_0|^2|\alpha|^2 + 7|\alpha|^2 = 0$

$\Rightarrow (|z_0|^2 - 1)(7|\alpha^2| - 1) = 0$

$\Rightarrow |z_0| = 1$ (rejected as r = 0) & $|\alpha| = \dfrac{1}{\sqrt{7}}$

32. $P^2 = [\alpha_{ij}]_{n \quad n}$

$\alpha_{ij} = \displaystyle\sum_{k=1}^{n} p_{ik}.p_{kj}$

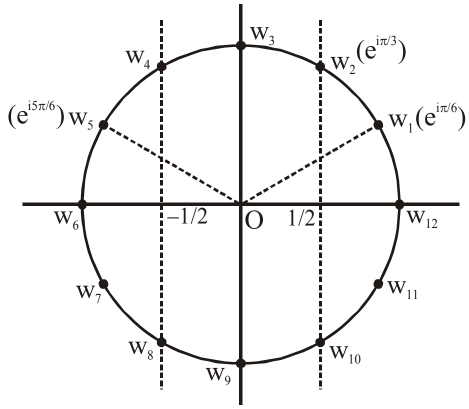
$= \displaystyle\sum_{k=1}^{n} \omega^{i+k}.\omega^{k+j} = \omega^{i+j}\sum_{k=1}^{n}\omega^{2k}$

$= \omega^{i+j}(\omega^2 + \omega^4 + \omega^6 + \ldots.. + \omega^{2n})$

If n is a multiple of 3 then $P^2 = \mathbf{0}$

$\Rightarrow$ n is not a multiple of 3

$\Rightarrow$ n can be 55, 58, 56

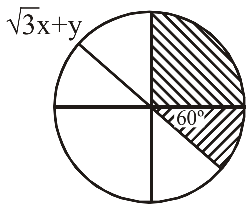
33. 

$z_1 = \{w_1, w_{11}, w_{12}\}$
$z_2 = \{w_5, w_6, w_7\}$

$\angle w_1Ow_5 = \dfrac{2\pi}{3}$ & $\angle w_1Ow_6 = \dfrac{5\pi}{6}$

Paragraph for Question 34 and 35

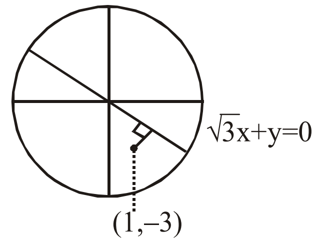
34. S_1 is interior of circle centred at (0,1) & radius = 4.



Re(z) > 0 is in I[st] & IV[th]quadrant.

$$\frac{\left(z-\left(1-i\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)} = \frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)}$$

$$= \frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)\left(1+i\sqrt{3}\right)}{2}$$



$I_m\left(S_2\right) = \sqrt{3}x + y > 0$

erpendicular distance from (1,–3) to the line is

$$P = \left|\frac{\sqrt{3}-3}{2}\right| = \left(\frac{3-\sqrt{3}}{2}\right)$$

35. **Ans. (B)**

Area of $S = \dfrac{\pi(4)^2}{4} + \dfrac{1}{2}(4)^2\dfrac{\pi}{3}$

$$\frac{8\pi}{3} + 4\pi = \frac{20\pi}{3}$$