UNIT # 10

VECTORS AND THREE DIMENSIONAL GEOMETRY

VECTOR

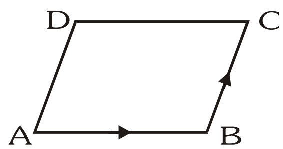
EXERCISE - 01 CHECK YOUR GRASP

4. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= 2\hat{i} - 2\hat{j} + 4\hat{k}$

$\overrightarrow{BD} = -\overrightarrow{AB} + \overrightarrow{BC}$

$= -4\hat{i} + 2\hat{j}$



Let Angle between $\overrightarrow{AC}$ & $\overrightarrow{BD}$ is θ

$\therefore \dfrac{\overrightarrow{AC}.\overrightarrow{BD}}{|\overrightarrow{AC}|\,|\overrightarrow{BD}|} = \cos\theta$

$\Rightarrow \cos\theta = \dfrac{-12}{4\sqrt{6}\sqrt{5}} = -\sqrt{\dfrac{3}{10}}$.

$\Rightarrow$ Acute angle between diagonals $= \cos^{-1}\sqrt{\dfrac{3}{10}}$

8. $\vec{a} = \hat{i} + \hat{j}$ & $\vec{b} = 2\hat{i} - \hat{k}$

$\vec{r} \times \vec{a} = \vec{b} \times \vec{a} \Rightarrow \left(\vec{r} - \vec{b}\right) \times \vec{a} = 0$

$\Rightarrow \vec{r} = \vec{b} + \lambda\vec{a}$ ...(i)

similarly $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$

$\Rightarrow \vec{r} = \vec{a} + \mu\vec{b}$ ...(ii)

Putting the vector $\vec{a}$ & $\vec{b}$ in (i) & (ii) and equating we get $2\hat{i} - \hat{k} + \lambda(\hat{i} + \hat{j}) = \hat{i} + \hat{j} + \mu(2\hat{i} - \hat{k})$

$\Rightarrow 2 + \lambda = 1 + 2\mu,\ \lambda = 1,\ \mu = 1$

$\therefore$ Point of intersection is $3\hat{i} + \hat{j} - \hat{k}$.

9. L. H. S = $\left(\lambda(\vec{a} + \vec{b}) \times \lambda^2\vec{b}\right).\lambda\vec{c}$

$= \lambda^4\left((\vec{a} + \vec{b}) \times \vec{b}\right).\vec{c} = \lambda^4[a\ b\ c]$

R.H.S. $= (\vec{a} \times (\vec{b} + \vec{c})).\vec{b} = [\vec{a}\ \vec{c}\ \vec{b}]$

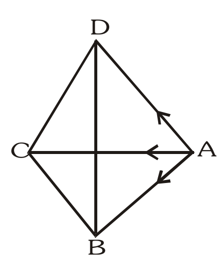
$\Rightarrow \lambda^4[a\ b\ c] = -[a\ b\ c]$

$\Rightarrow \lambda^4 = -1$ which is not possible.

11. $\overrightarrow{AD} = -2\hat{i} + 2\hat{j} - \hat{k}$

$\overrightarrow{AC} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\overrightarrow{AB} = 3\hat{j} + 4\hat{k}$



$$\vec{n}_1 = \overrightarrow{AD} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 6\hat{i} + 3\hat{j} - 6\hat{k}$$

$$= 3\left(2\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$\vec{n}_2 = \overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{vmatrix} = 2\hat{i} - 4\hat{j} + 3\hat{k}$$

$$|\vec{n}_1 \times \vec{n}_2| = 3\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 2 & -4 & 3 \end{vmatrix} = 3(-5\hat{i} - 10\hat{j} - 10\hat{k})$$

$$\sin\theta = \frac{5}{\sqrt{29}} \qquad \left(\sin\theta = \frac{|\vec{n}_1 \times \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}\right)$$

14. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{i} \times (\vec{r} \times \hat{i}) + \hat{j} \times (\vec{r} \times \hat{j}) + \hat{k} \times (\vec{r} \times \hat{k})$

$= (\hat{i}.\hat{i})\,\vec{r} - (\hat{i}.\vec{r})\hat{i} + (\hat{j}.\hat{j})\vec{r} - (\hat{j}.\vec{r})\hat{j} + (\hat{k}.\hat{k})\vec{r} - (\hat{k}.\vec{r})\hat{k}$

$= 3\vec{r} - \hat{i}(\hat{i}.(x\hat{i} + y\hat{j} + z\hat{k})) - \hat{j}(\hat{j}.(x\hat{i} + y\hat{j} + z\hat{k}))$

$- \hat{k}(\hat{k}.(x\hat{i} + y\hat{j} + z\hat{k}))$

$= 3\vec{r} - (x\hat{i} + y\hat{j} + z\hat{k}) = 3\vec{r} - \vec{r} = 2\vec{r}$.

20. $(a - b)\vec{x} + (b - c)\vec{y} + (c - a)(\vec{x} \times \vec{y}) = 0$

As $\vec{x}, \vec{y}$ & $(\vec{x} \times \vec{y})$ are non zero, non coplanar vectors, then

$a - b = b - c = c - a = 0 \quad \Rightarrow \quad a = b = c$

Hence ΔABC is an equilateral triangle.

Hence, acute angled triangle.

23. $\vec{a}, \vec{b}, \vec{c}$ are unit vector mutually perpendicular to each other then angle between $\vec{a} + \vec{b} + \vec{c}$ & $\vec{a}$ is given by

$$\cos\theta = \frac{(\vec{a} + \vec{b} + \vec{c}).\vec{a}}{|\vec{a} + \vec{b} + \vec{c}|\,|\vec{a}|} = \frac{\left(\vec{a} + \vec{b} + \vec{c}\right).\vec{a}}{\sqrt{\vec{a}^2 + \vec{b}^2 + \vec{c}^2}\,|\vec{a}|}$$

$$= \frac{|\vec{a}|}{\sqrt{\vec{a}^2 + \vec{b}^2 + \vec{c}^2}}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\sqrt{2}$$

24. $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

But $\vec{b}.\vec{c} \neq 0$, $\vec{a}.\vec{b} \neq 0$

$\Rightarrow$ $\vec{a}$ & $\vec{c}$ must be parallel.

25. $\vec{a} \times (\vec{b} \times \vec{c}) = \dfrac{\vec{b} + \vec{c}}{\sqrt{2}}$

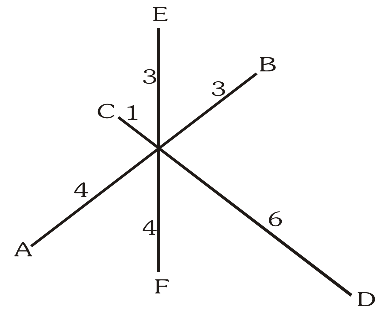
$$\Rightarrow (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \frac{\vec{b}}{\sqrt{2}} + \frac{\vec{c}}{\sqrt{2}}$$

$$\Rightarrow \left(\vec{a}.\vec{c} - \frac{1}{\sqrt{2}}\right)\vec{b} - \left(\vec{a}.\vec{b} + \frac{1}{\sqrt{2}}\right)\vec{c} = 0$$

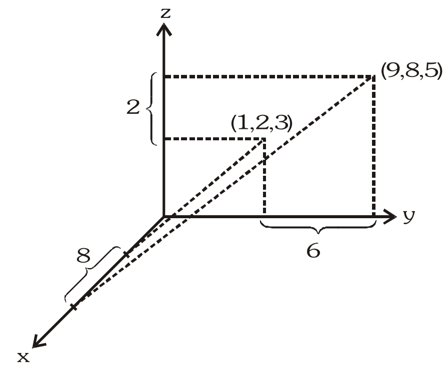
$$\therefore \quad \vec{a}.\vec{b} = \frac{-1}{\sqrt{2}} \ \& \ \vec{a}.\vec{c} = \frac{1}{\sqrt{2}}$$

angle between $\vec{a}$ & $\vec{b}$ $= \cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right) = \dfrac{3\pi}{4}$

26. $\dfrac{3\vec{a} + 4\vec{b}}{7} = \dfrac{6\vec{c} + \vec{d}}{7} = \dfrac{4\vec{e} + 3\vec{f}}{7} = \dfrac{\vec{x}}{7}$



30.



Hence, edge length of the parallelopiped

$|x_2 - x_1| = 8$

$|y_2 - y_1| = 6$

$|z_2 - z_1| = 2$

EXERCISE - 02 — BRAIN TEASERS

2. $\vec{r}_1 = \vec{a} - \vec{b} + \vec{c}$ ...(i)

$\vec{r}_2 = \vec{b} + \vec{c} - \vec{a}$ ...(ii)

$\vec{r}_3 = \vec{c} + \vec{a} + \vec{b}$ ...(iii)

$\vec{r} = 2\vec{a} - 3\vec{b} + 4\vec{c}$

If $\vec{r} = \lambda_1\vec{r}_1 + \lambda_2\vec{r}_2 + \lambda_3\vec{r}_3$

then $2\vec{a} - 3\vec{b} + 4\vec{c}$

$= (\lambda_1 - \lambda_2 + \lambda_3)\ \vec{a} + (\lambda_2 - \lambda_1 + \lambda_3)\vec{b} + (\lambda_1 + \lambda_2 + \lambda_3)\vec{c}$

$\Rightarrow \lambda_1 + \lambda_3 - \lambda_2 = 2$ ...(iv)

$\lambda_2 + \lambda_3 - \lambda_1 = -3$ ...(v)

$\lambda_1 + \lambda_2 + \lambda_3 = 4$ ...(vi)

Solving (iv) (v) & (vi) we get

$\lambda_2 = 1$; $\lambda_1 = 7/2$; $\lambda_3 = -1/2$

Now check options

5. $\vec{p} \times ((\vec{x} - \vec{q}) \times \vec{p}) + \vec{q} \times ((\vec{x} - \vec{r}) \times \vec{q})$

$+ \vec{r} \times ((\vec{x} - \vec{p}) \times \vec{r}) = 0$

$= (\vec{p}.\vec{p})(\vec{x} - \vec{q}) - (\vec{p}.(\vec{x} - \vec{q})\vec{p}) + (\vec{q}.\vec{q})(\vec{x} - \vec{r})$

$-(\vec{q}.(\vec{x} - \vec{r})\vec{q}) + \vec{r}.\vec{r}(\vec{x} - \vec{p}) - (\vec{r}.(\vec{x} - \vec{p})\vec{r}) = 0$

$\Rightarrow \lambda^2\Sigma(\vec{x} - \vec{q}) - \Sigma(\vec{p}.\vec{x})\vec{p} = 0$,

where $\vec{p}.\vec{p} = \vec{q}.\vec{q} = \vec{r}.\vec{r} = \lambda^2$

(Since $\vec{p}.\vec{q} = \vec{q}.\vec{r} = \vec{r}.\vec{p} = 0$)

$\Rightarrow \lambda^2\left[3\vec{x} - (\vec{p} + \vec{q} + \vec{r})\right] - \Sigma\{(\vec{p}.\vec{x})\vec{p}\} = 0$

$[\Sigma(\vec{p}.\vec{x})\vec{p} = \vec{x}\lambda^2]$

$2\vec{x} = \vec{p} + \vec{q} + \vec{r} \Rightarrow \vec{x} = \dfrac{\vec{p} + \vec{q} + \vec{r}}{2}$.

6. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

it makes equal angle with

$\dfrac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}),\ \dfrac{1}{5}(-4\hat{i} - 3\hat{k}), \hat{j}$ then

$$\frac{x - 2y + 2z}{3} = \frac{-4x - 3z}{5} = y$$

$4x + 5y + 3z = 0$ ...(i)

$x - 5y + 2z = 0$ ...(ii)

from (i) & (ii)

$x = -z$ & $x = -5y$

$\vec{a} = x\left(\hat{i} - \dfrac{1}{5}\hat{j} - \hat{k}\right)$.

7. $(\vec{d} + \vec{a}).[\vec{a} \times \{(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}\}]$

$= (\vec{d} + \vec{a}) \cdot [(\vec{b}.\vec{d})(\vec{a} \times \vec{c}) - (\vec{b}.\vec{c})(\vec{a} \times \vec{d})]$

$= [\vec{d}\ \vec{a}\ \vec{c}]\ (\vec{b}.\vec{d}) - 0 = [\vec{d}\ \vec{a}\ \vec{c}]\ (\vec{b}.\vec{d})$.

8. $[(\ell\vec{a} + m\vec{b} + n\vec{c})(\ell\vec{b} + m\vec{c} + n\vec{a})(\ell\vec{c} + m\vec{a} + n\vec{b})] = 0$

Let $\vec{n}_1 = (\ell\vec{b} + m\vec{c} + n\vec{a}) \quad (\ell\vec{c} + m\vec{a} + n\vec{b})$

$= (\ell^2 - mn)(\vec{b} \times \vec{c}) + (\vec{a} \times \vec{b})(n^2 - \ell m) + (m^2 - \ell n)$

Now $(\ell\vec{a} + m\vec{b} + n\vec{c}).\vec{n}_1$

$\Rightarrow [abc]\ (\ell^3 - \ell mn + m^3 - \ell mn + n^3 - \ell mn) = 0$

$\Rightarrow [abc]\ (\ell^3 + m^3 + n^3 - 3\ell mn) = 0$

$\Rightarrow \ell + m + n = 0.$

12. (A) $\vec{a} \times [a \times (\vec{a} \times \vec{b})]$

$= \vec{a} \times [(a.b)\vec{a} - (\vec{a}.\vec{a})\vec{b}] = 0 - (\vec{a})^2(\vec{a} \times \vec{b})$. False

(B) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$$\left.\begin{array}{l} \vec{v}.\vec{a} = 0 \\ \vec{v}.\vec{b} = 0 \\ \vec{v}.\vec{c} = 0 \end{array}\right\} \Rightarrow \vec{v}.(\vec{a} + \vec{b} + \vec{c}) = 0$$

But $\vec{a} + \vec{b} + \vec{c} \neq 0 \Rightarrow \vec{v} = 0$. i.e. null vector which is true

(C) $\vec{a} \times \vec{b}$ & $\vec{c} \times \vec{d}$ are perpendicular

so $(\vec{a} \times \vec{b}) \quad (\vec{c} \times \vec{d}) \neq 0$. False

(D) $a' = \dfrac{\vec{b} \times \vec{c}}{[a\,b\,c]}$, $b' = \dfrac{\vec{c} \times \vec{a}}{[a\,b\,c]}$, $c' = \dfrac{\vec{a} \times \vec{b}}{[a\,b\,c]}$

is valid only if $\vec{a}, \vec{b}, \vec{c}$ are non coplanar, hence false.

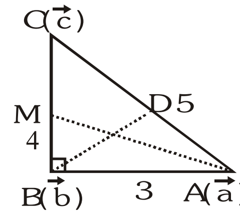
14. $\vec{b}.(\vec{a} + \vec{c}) = \vec{b}.\vec{b} + \vec{a}.\vec{c}$

$\vec{b}.(\vec{a} - \vec{b}) + \vec{c}.(\vec{b} - \vec{a}) = 0$

$(\vec{b} - \vec{c}).(\vec{a} - \vec{b}) = 0$

$\Rightarrow$ angle between

$\vec{b} - \vec{c}$ & $\vec{a} - \vec{b} = 90°$



Let B be the origin & A $(3\hat{i})$, C $(4\hat{j})$

then $M(2\hat{j})$ & $D\left(\dfrac{3\hat{i} + 4\hat{j}}{2}\right)$

$\therefore$ Angle between $\overrightarrow{BD}$ & $\overrightarrow{AM}$

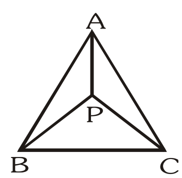
$$\cos\theta = \frac{(2\hat{j} - 3\hat{i}).\left(\dfrac{3\hat{i} + 4\hat{j}}{2}\right)}{\sqrt{13} \times \dfrac{5}{2}} = -\frac{1}{5\sqrt{13}}$$

$$\cos\theta = -\frac{1}{5\sqrt{13}}.$$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Match the column :

1. (A) If P is a point inside the triangle such that area($\Delta PAB + \Delta PBC + \Delta PCA$) = area ($\Delta ABC$) Then P is centroid.



(B) $\overrightarrow{V} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}$

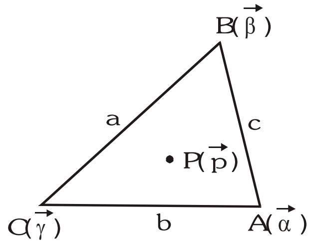
$0 = \vec{a} - \vec{p} + \vec{b} - \vec{p} + \vec{c} - \vec{p}$

$\vec{p} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$ which is centroid.

(C) $\vec{P} = (BC)\overrightarrow{PA} + (CA)\overrightarrow{PB} + (AB)\overrightarrow{PC} = 0$

$a(\vec{\alpha} - \vec{p}) + b(\vec{\beta} - \vec{p}) + c(\vec{\gamma} - \vec{p}) = 0$

$\Rightarrow \vec{p} = \dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a + b + c}$



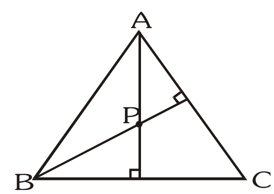
which is incentre.

(D) From fig.

$\overrightarrow{PA} \cdot \overrightarrow{CB} = 0$

$\overrightarrow{PB}.\overrightarrow{AC} = 0$

$\Rightarrow$ P is orthocentre.



Assertion & Reason :

2. **Statement - I :** $A(\vec{a})$ & $B(\vec{b})$

$\overrightarrow{PA}.\overrightarrow{PB} \leq 0$, then locus of P is sphere having diameter $|(\vec{a} - \vec{b})|$

volume $= \dfrac{4}{3}\pi\left|\dfrac{\vec{a} - \vec{b}}{2}\right|^3 = \dfrac{\pi}{6}|\vec{a} - \vec{b}|^2 \cdot |\vec{a} - \vec{b}|$

$= \dfrac{\pi}{6}(\vec{a}^2 + \vec{b}^2 - 2\vec{a}.\vec{b})|\vec{a} - \vec{b}|$

Hence true.

Statement - II : Diameter of sphere subtend acute angle at point P then point P moves out side the sphere having radius r.

5. **Statement - I :**

$\vec{a} = \hat{i}, \vec{b} = \hat{j}$ & $\vec{c} = \hat{i} + \hat{j}$

$\vec{c} = \vec{a} + \vec{b}$ linearly dependent

$\vec{a}$ & $\vec{b}$ are linearly independent

Hence true.

Statement - II :

$\vec{a}$ & $\vec{b}$ are linearly dependent

$\vec{a} = t\vec{b}$

then $\vec{c} = \lambda\vec{a} + \mu\vec{b}$ which is linearly dependent.

Comprehension # 2

Vector $\vec{p} = \hat{i} + \hat{j} + \hat{k}$, $\vec{q} = 2\hat{i} + 4\hat{j} - \hat{k}$,

$\vec{r} = \hat{i} + \hat{j} + 3\hat{k}$

1. (A) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix} = 13 - 7 - 2 = 4 \neq 0$

Hence non coplanar; so linearly independent

(B) In triangle, let length of sides of triangle are a, b, c then triangle is formed if sum of two sides is greater than the third side. Check yourself.

(C) $(\vec{q} - \vec{r}).\vec{p}$

$= (i + 3j - 4k) . (i + j + k) = 1 + 3 - 4 = 0$

Hence true.

2. $((\vec{p} \times \vec{q}) \times \vec{r}) = u\vec{p} + v\vec{q} + w\vec{r}$

$(\vec{p}.\vec{r})\vec{q} - (\vec{q}.\vec{r})\vec{p} = u\vec{p} + v\vec{q} + w\vec{r}$

By solving $\vec{p}.\vec{r}$ & $\vec{q}.\vec{r}$, we get

$5\vec{q} - 3\vec{p} + 0\vec{r} = u\vec{p} + v\vec{q} + w\vec{r}$

compare

$u + v + w = 5 - 3 + 0 = 2.$

3. $\vec{s}$ is unit vector

$(\vec{p}.\vec{s})\ (\vec{q} \times \vec{r}) + (\vec{q}.\vec{s})\ (\vec{r} \times \vec{p}) + \vec{r}.\vec{s}\ (\vec{p} \times \vec{q})$

$$\vec{q} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix} = 13\hat{i} - 7\hat{j} - 2\hat{k}$$

$$\vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -2\hat{i} + 2\hat{j}$$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & -1 \end{vmatrix} = -5\hat{i} + 3\hat{j} + 2\hat{k}$$

Let $\vec{s} = \hat{i}$

Putting the value we get

$13\hat{i} - 7\hat{j} - 2\hat{k} + 2(-2\hat{i} + 2\hat{j}) + (-5\hat{i} + 3\hat{j} + 2\hat{k})$

$= 13\hat{i} - 7\hat{j} - 2\hat{k} - 4\hat{i} + 4\hat{j} - 5\hat{i} + 3\hat{j} + 2\hat{k}$

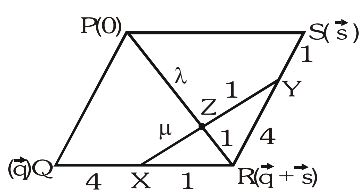
$= 4\hat{i} + 0\hat{j} + 0\hat{k} = 4\hat{i}$

Magnitude = 4.

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

5. $\overrightarrow{QX} = 4\overrightarrow{XR}$

$\overrightarrow{RY} = 4\overrightarrow{YS}$



Let $\vec{p}$ be origin

& R $(\vec{q} + \vec{s})$

from figure

P.V. of X $= \dfrac{4(\vec{q} + \vec{s}) + \vec{q}}{5} = \dfrac{5\vec{q} + 4\vec{s}}{5}$

P.V. of Y $= \dfrac{4\vec{s} + \vec{q} + \vec{s}}{5} = \dfrac{5\vec{s} + \vec{q}}{5}$

Now Let Z divides PR in ratio $\lambda : 1$

Now Let Z divides XY in ratio $\mu : 1$

P.V. of Z $= \dfrac{\lambda(\vec{q} + \vec{s})}{\lambda + 1}$ (from PR)

P.V. of Z $= \dfrac{\dfrac{\mu(5\vec{s} + \vec{q})}{5} + \dfrac{5\vec{q} + 4\vec{s}}{5}}{\mu + 1}$ (from XY)

equating both Z then we get

$\dfrac{\lambda}{\lambda + 1} = \dfrac{\mu + 5}{5(\mu + 1)}$ ....(i)

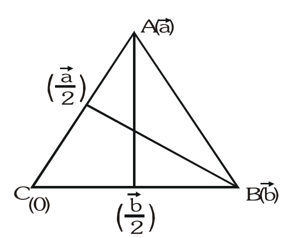
$\dfrac{\lambda}{\lambda + 1} = \dfrac{5\mu + 4}{5(\mu + 1)}$ ......(ii)

from (i) & (ii), $\mu = \dfrac{1}{4}$ & $\lambda = \dfrac{21}{4}$

So P.V. of Z $= \dfrac{\frac{21}{4}}{\frac{21}{4} + 1}(\vec{q} + \vec{s})$

$= \dfrac{21}{25}(\vec{q} + \vec{s}) = \dfrac{21}{25}\overrightarrow{PR}$

10. Let origin be C



Given $\left|\vec{a} - \dfrac{\vec{b}}{2}\right| = \left|\vec{b} - \dfrac{\vec{a}}{2}\right|$ (medians are equal)

$\Rightarrow \vec{a}^2 + \dfrac{\vec{b}^2}{4} - \vec{a}.\vec{b} = \vec{b}^2 + \dfrac{\vec{a}^2}{4} - \vec{a}.\vec{b}$

$$\frac{3\vec{a}^2}{4} = \frac{3}{4}\vec{b}^2 \qquad \Rightarrow |\vec{a}| = |\vec{b}|$$

12. Let $|\vec{u}| = \lambda$

$$\vec{u} = \frac{\lambda}{2}\ (\tilde{i} + \sqrt{3}\,\tilde{j})$$

Given $\left|\frac{\lambda}{2}(\tilde{i} + \sqrt{3}\,\tilde{j}) - \tilde{i}\right|^2 = \lambda\left|\frac{\lambda}{2}(\tilde{i} + \sqrt{3}\,\tilde{j}) - 2\tilde{i}\right|$

$$\left(\left(\frac{\lambda}{2} - 1\right)^2 + \frac{3\lambda^2}{4}\right)^2 = \lambda^2\left(\left(\frac{\lambda - 4}{2}\right)^2 + \frac{3\lambda^2}{4}\right)$$

$(4\lambda^2 - 4\lambda + 4)^2 = 16\lambda^2\ (\lambda^2 - 2\lambda + 4)$

$(\lambda^2 - \lambda + 1)^2 = \lambda^2\ (\lambda^2 - 2\lambda + 4)$

solving we get $\lambda = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

But $\lambda > 0$

$\Rightarrow\ \lambda = \sqrt{2} - 1$

$\therefore\ a = 2,\ \ b = 1$

14. $|\vec{r} + b\vec{s}|$ is minimum

Let $f(b) = \sqrt{\vec{r}^2 + b^2\vec{s}^2 + 2\vec{r}.b\vec{s}}$

for maxima & minima

$$f'(b) = \frac{2b\vec{s}^2 + 2\vec{r}.\vec{s}}{\sqrt{\vec{r}^2 + b^2\vec{s}^2 + 2b\vec{r}.\vec{s}}} = 0$$

$$b = -\frac{\vec{r}.\vec{s}}{\vec{s}^2}$$

$$|b\vec{s}|^2 + |\vec{r} + b\vec{s}|^2 = b^2\vec{s}^2 + \vec{r}^2 + b^2\vec{s}^2 + 2b\,\vec{r}.\vec{s}$$

$$= 2b^2\vec{s}^2 + \vec{r}^2 - 2b^2\vec{s}^2 = |\vec{r}|^2$$

17. O (0,0), A(1,0) & B (–1, 0)

Let P (x,y)

$\overrightarrow{PA} = (1 - x)\,\tilde{i} - y\tilde{j}$

$\overrightarrow{PB} = -(1 + x)\,\tilde{i} - y\tilde{j}$

$\overrightarrow{PA} \cdot \overrightarrow{PB} + 3\ \overrightarrow{OA}.\overrightarrow{OB} = 0$

$\Rightarrow (x^2 - 1) + y^2 - 3 = 0$

$x^2 + y^2 = 4 \qquad \ldots (1)$

$$|\overrightarrow{PA}||\overrightarrow{PB}| = \sqrt{(x-1)^2 + y^2}\ \sqrt{(x+1)^2 + y^2}$$

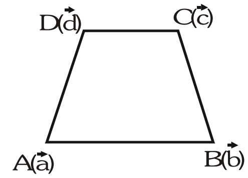
$$= \sqrt{5 - 2x} \cdot \sqrt{5 + 2x}$$

$$= \sqrt{25 - 4x^2}\ , \qquad x \in (-2,2) \text{ (from (1))}$$

so M = 5, m = 3

$\Rightarrow M^2 + m^2 = 25 + 9 = 34$

20.



In cyclic quadrilateral

tanA + tan C = 0

$$\Rightarrow \frac{|\overrightarrow{AB} \times \overrightarrow{AD}|}{\overrightarrow{AB}.\overrightarrow{AD}} + \frac{|\overrightarrow{CB} \times \overrightarrow{CD}|}{\overrightarrow{CB}.\overrightarrow{CD}} = 0$$

$$\Rightarrow \frac{|(\vec{b} - \vec{a}) \times (\vec{d} - \vec{a})|}{(\vec{b} - \vec{a}).(\vec{d} - \vec{a})} + \frac{|(\vec{b} - \vec{c}) \times (\vec{d} - \vec{c})|}{(\vec{b} - \vec{c}).(\vec{d} - \vec{c})} = 0$$

$$\Rightarrow \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a}).(\vec{d} - \vec{a})} + \frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c}).(\vec{d} - \vec{c})} = 0$$

21. $\vec{a} = \sqrt{3}\,i - \tilde{j}\ ,\ \vec{b} = \dfrac{1}{2}\tilde{i} + \dfrac{\sqrt{3}}{2}\tilde{j}$

$\Rightarrow\ \vec{a}.\vec{b} = 0$

$\vec{x}.\vec{y} = 0$ given

$(\vec{a} + (q^2 - 3)\vec{b}) \cdot (-p\vec{a} + q\vec{b}) = 0$

$$\Rightarrow p = \frac{q(q^2 - 3)}{4} = f(q)$$

for monotonocity

$p' = 3q^2 - 3$

if $p' < 0$ then f(q) is decreasing

$\Rightarrow (q - 1)\ (q + 1) < 0$

$\Rightarrow -1 < q < 1$

Decreasing for $q \in (-1, 1),\ q \neq 0$

24. $|(\vec{a}.\vec{d})(\vec{b} \times \vec{c}) + (\vec{b}.\vec{d})(\vec{c} \times \vec{a}) + (\vec{c}.\vec{d})(\vec{a} \times \vec{b})|$

$|(\vec{a}.\vec{d})(\vec{b} \times \vec{c}) - (\vec{c}.\vec{d})(\vec{b} \times \vec{a}) + (\vec{b}.\vec{d})(\vec{c} \times \vec{a})|$

$|\vec{b} \times [(\vec{a}.\vec{d})\vec{c} - (\vec{c}.\vec{d})\vec{a}] + (\vec{b}.\vec{d})(\vec{c} \times \vec{a})|$

$|\vec{b} \times \{(\vec{a} \times \vec{c}) \times \vec{d}] + (\vec{b}.\vec{d})(\vec{c} \times \vec{a})|$

$= |(\vec{b}.\vec{d})(\vec{a} \times \vec{c}) - \{\vec{b}.(\vec{a} \times \vec{c})\}\vec{d} - (\vec{b}.\vec{d})(\vec{a} \times \vec{c})|$

$= |[\vec{b}\ \vec{a}\ \vec{c}]\vec{d}| = [\vec{b}\ \vec{a}\ \vec{c}]\,|\vec{d}| \qquad \because\ |\vec{d}| = 1$

$= [\vec{b}\ \ \vec{a}\ \ \vec{c}]$ Proved.

28. $\vec{x}\times\vec{a}+(\vec{x}.\vec{b})\vec{a}=\vec{c}$ ....(i)

taking cross product with $\vec{b}$:

$(\vec{x}\times\vec{a})\times\vec{b}+(\vec{x}.\vec{b})(\vec{a}\times\vec{b})=\vec{c}\times\vec{b}$

$(\vec{x}.\vec{b})\vec{a}-(\vec{a}.\vec{b})\vec{x}+(\vec{x}.\vec{b})(\vec{a}\times\vec{b})=\vec{c}\times\vec{b}$ ....(ii)

Now taking dot product with $\vec{a}$ in (i)

$(\vec{x}.\vec{b})a^2=\vec{a}.\vec{c}$

$$\vec{x}.\vec{b}=\frac{\vec{a}.\vec{c}}{a^2}$$

$$\frac{(\vec{a}.\vec{c})}{a^2}\vec{a}-(\vec{a}.\vec{b})\vec{x}+\frac{\vec{a}.\vec{c}}{a^2}(\vec{a}\times\vec{b})=\vec{c}\times\vec{b}$$

$$\frac{1}{(\vec{a}.\vec{b})}\left[\frac{(\vec{a}.\vec{c})}{a^2}\vec{a}+\frac{\vec{a}.\vec{c}}{a^2}\left(\vec{a}\times\vec{b}\right)-\vec{c}\times\vec{b}\right]=\vec{x}$$

$$\vec{x}=\frac{1}{(\vec{a}.\vec{b})}\left[\frac{\vec{a}.\vec{c}}{a^2}\left(\vec{a}-\vec{b}\times\vec{a}\right)+\vec{b}\times\vec{c}\right]$$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

2. $\vec{a}=a_1\tilde{i}+a_2\tilde{j}+a_3\tilde{k}$

$\vec{b}=b_1\tilde{i}+b_2\tilde{j}+b_3\tilde{k}$

$\vec{c}=c_1\tilde{i}+c_2\tilde{j}+c_3\tilde{k}$

$[\vec{a}\,\vec{b}\,\vec{c}]$ is written as $\begin{vmatrix}\vec{a}.\tilde{i} & \vec{a}.\tilde{j} & \vec{a}.\tilde{k}\\ \vec{b}.\tilde{i} & \vec{b}.\tilde{j} & \vec{b}.\tilde{k}\\ \vec{c}.\tilde{i} & \vec{c}.\tilde{j} & \vec{c}.\tilde{k}\end{vmatrix}$

Now $\{(n\vec{a}+\vec{b})\times(n\vec{b}+\vec{c})\}.(n\vec{c}+\vec{a})$

$=\{n^2(\vec{a}\times\vec{b})+n(\vec{a}\times\vec{c})+\vec{b}\times\vec{c}\}\cdot(n\vec{c}+\vec{a})$

$=n^3[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{c}\,\vec{a}]$

$=(n^3+1)\,[\vec{a}\,\vec{b}\,\vec{c}]$

5. Given $\overrightarrow{OP_{n-1}}+\overrightarrow{OP_{n+1}}=\frac{3}{2}\overrightarrow{OP_n}$ n = 2,3

(a) Let P_1 & P_2 be $\left(t_1,\frac{1}{t_1}\right)$ & $\left(t_2,\frac{1}{t_2}\right)$

for n = 2

$$\overrightarrow{OP_1}+\overrightarrow{OP_3}=\frac{3}{2}\overrightarrow{OP_2}$$

$$\Rightarrow \overrightarrow{OP_3}=\frac{3}{2}\left(t_2\tilde{i}+\frac{1}{t_2}\tilde{j}\right)-t_1i-\frac{1}{t_1}\tilde{j}$$

or $$\overrightarrow{OP_3}=\left(\frac{3}{2}t_2-t_1\right)\tilde{i}+\left(\frac{3}{2t_2}-\frac{1}{t_1}\right)\tilde{j}$$

Point $P_3=\left(\frac{3t_2-2t_1}{2},\frac{3t_1-2t_2}{2t_1t_2}\right)$

which does not lie on xy = 1

(b) Let P_1 & P_3 on circle $x^2+y^2=1$ are $(\cos\alpha,\sin\alpha)$, $(\cos\beta,\sin\beta)$

For n = 2, $\overrightarrow{OP_1}+\overrightarrow{OP_3}=\frac{3}{2}\overrightarrow{OP_2}$

$$\overrightarrow{OP_2}=\frac{2}{3}\left\{(\cos\alpha\,\tilde{i}+\sin\alpha\,\tilde{j})+(\cos\beta\,\tilde{i}+\sin\beta\,\tilde{j})\right\}$$

$$\overrightarrow{OP_2}=\frac{2}{3}\left\{(\cos\alpha+\cos\beta)\tilde{i}+(\sin\alpha+\sin\beta)\tilde{j}\right\}$$

As P_2 lies on the circle then

$$\left|\overrightarrow{OP_2}\right|=1$$

$$\frac{4}{9}\left\{(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2\right\}=1$$

$$2+2\cos(\alpha-\beta)=\frac{9}{4}$$

$$\Rightarrow \cos(\alpha-\beta)=\frac{1}{8}$$

$$\overrightarrow{OP_4}=\frac{3}{2}\overrightarrow{OP_3}-\frac{2}{3}\left(\overrightarrow{OP_1}+\overrightarrow{OP_3}\right)$$

$$=\frac{5}{6}\overrightarrow{OP_3}-\frac{2}{3}\overrightarrow{OP_1}$$

$$=\left(\frac{5}{6}\cos\alpha-\frac{2}{3}\cos\beta\right)\tilde{i}+\left(\frac{5}{6}\sin\alpha-\frac{2}{3}\sin\beta\right)\tilde{j}$$

$$\left|\overrightarrow{OP_4}\right|^2=\frac{25}{36}+\frac{4}{9}-2.\frac{5}{6}.\frac{2}{3}\cos\left(\alpha-\beta\right)=1$$

$\Rightarrow P_4$ lies on $x^2+y^2=1$

9. $\vec{x}+\vec{c}\times\vec{y}=\vec{a}$ ......(i)

$\vec{y}+\vec{c}\times\vec{x}=\vec{b}$ .....(ii)

$\Rightarrow \vec{y}=\vec{b}-\vec{c}\times\vec{x}$ put in (i)

$\vec{x}+\vec{c}\times\vec{b}-\vec{c}\;(\vec{c}\times\vec{x})=\vec{a}$

$\vec{x}-(\vec{c}.\vec{x})\,\vec{c}+(\vec{c}.\vec{c})\vec{x}=\vec{a}-\vec{c}\times\vec{b}$

$(1+c^2)\,\vec{x}=\vec{a}-\vec{c}\times\vec{b}+(\vec{c}.\vec{x})\,\vec{c}$ ... (iii)

Taking both side dot product with $\vec{c}$ in equation (i)

We get $\vec{x}.\vec{c}=\vec{a}.\vec{c}$, (put in (iii)

$$\vec{x} = \frac{a + (\vec{a}.\vec{c})\vec{c} + \vec{b} \times \vec{c}}{1 + \vec{c}^2}$$

Putting in (ii), we get $\vec{y} = \dfrac{\vec{b} + (\vec{c}.\vec{b})\vec{c} + \vec{a} \times \vec{c}}{1 + (\vec{c})^2}$

10. $\vec{\alpha} = \hat{i} + a\hat{j} + a^2\hat{k}$

$\vec{\beta} = \hat{i} + b\hat{j} + b^2\hat{k}$

$\vec{\gamma} = \hat{i} + c\hat{j} + c^2\hat{k}$

$\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are non coplanar

$$\therefore \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

$\Rightarrow$ (a–b) (b–c) (c – a) $\neq 0$ $\Rightarrow$ $a \neq b \neq c$

If α_1, β_1 & γ_1 are coplanar

$$\text{Then } \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & b_1 & b_1^2 \\ 1 & c_1 & c_1^2 \end{vmatrix} = 0 \quad \Rightarrow a_1 = b_1 = c_1$$

$$\text{Given } \begin{vmatrix} (a_1 - a)^2 & (a_1 - b)^2 & (a_1 - c)^2 \\ (b_1 - a)^2 & (b_1 - b)^2 & (b_1 - c)^2 \\ (c_1 - a)^2 & (c_1 - b)^2 & (c_1 - c)^2 \end{vmatrix} = 0$$

$\Rightarrow R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$, we get

$$(a_1-b_1)(b_1-c_1)\begin{vmatrix} a_1 + b_1 - 2a & a_1 + b_1 - 2b & a_1 + b_1 - 2c \\ b_1 + c_1 - 2a & b_1 + c_1 - 2b & b_1 + c_1 - 2c \\ (c_1 - a)^2 & (c_1 - b)^2 & (c_1 - c)^2 \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\Rightarrow (a_1-b_1)\ (b_1-c_1)\begin{vmatrix} a_1 - c_1 & a_1 - c_1 & a_1 - c_1 \\ b_1 + c_1 - 2a & b_1 + c_1 - 2b & b_1 + c_1 - 2c \\ (c_1 - a)^2 & (c_1 - b)^2 & (c_1 - c^2) \end{vmatrix} = 0$$

$\Rightarrow (a_1-b_1)\ (b_1 - c_1)\ (c_1 - a_1)$

$$\begin{vmatrix} 1 & 1 & 1 \\ b_1 + c_1 - 2a & b_1 + c_1 - 2b & b_1 + c_1 - 2c \\ (c_1 - a)^2 & (c_1 - b)^2 & (c_1 - c)^2 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$\Rightarrow (a_1 - b_1)\ (b_1-c_1)\ (c_1 - a_1)$

$$\underbrace{\begin{vmatrix} 0 & 0 & 1 \\ 2(b-a) & 2(c-b) & b_1 + c_1 - 2c \\ a^2 - b^2 - 2c_1(a-b) & b^2 - c^2 - 2c_1(b-c) & (c_1 - c)^2 \end{vmatrix}}_{\Delta} = 0$$

$(a_1-b_1)\ (b_1 - c_1)\ (c_1 - a_1)\ \Delta = 0$

$\Rightarrow (a_1 - b_1)\ (b_1 - c_1)\ (c_1 - c_1) = 0 \qquad [\Delta \neq 0]$

$\Rightarrow a_1 = b_1 = c_1$

$\Rightarrow \vec{\alpha}_1, \vec{\beta}_1, \vec{\gamma}_1$ are coplanar

11. $\overrightarrow{OP} = \hat{i} + 2\hat{j} + 2\hat{k}$

after rotation of $\overrightarrow{OP}$, let new vector is $\overrightarrow{OP}\,'$

Now $\overrightarrow{OP}$, $\hat{i}$, $\overrightarrow{OP}\,'$ will be coplanar

So $\overrightarrow{OP}\,' = \left|\overrightarrow{OP}\right| \dfrac{(\overrightarrow{OP} \times \hat{i}) \times \overrightarrow{OP}}{\left|(\overrightarrow{OP} \times \hat{i}) \times \overrightarrow{OP}\right|} \quad \left[\because \left|\overrightarrow{OP}\right| = \left|\overrightarrow{OP}\,'\right|\right]$

But $(\overrightarrow{OP} \times \hat{i}) \times \overrightarrow{OP} = 8\hat{i} - 2\hat{j} - 2\hat{k}$

$$\Rightarrow \overrightarrow{OP}\,' = \frac{3(8\hat{i} - 2\hat{j} - 2\hat{k})}{2 \times 3\sqrt{2}}$$

$$\text{or } \overrightarrow{OP}\,' = \frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$$

EXERCISE - 05 [A] — JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

6. We have,

$\vec{u}.\vec{n} = 0$ and $\vec{v}.\vec{n} = 0$

$\Rightarrow \quad \vec{n} \perp \vec{u}$ and $\vec{n} \perp \vec{v}$

$$\Rightarrow \quad \vec{n} = \pm \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$$

Now, $\vec{u} \times \vec{v} = (\hat{i} + \hat{j}) \times (\hat{i} - \hat{j}) = -2\hat{k}$

$\therefore \quad \vec{n} = \pm \hat{k}$

Hence, $|\vec{w}.\vec{n}| = |(\hat{i} + 2\hat{j} + 3\hat{k}).(\pm\hat{k})| = 3$

7. We have,

$\vec{F}$ = Total force = $7\hat{i} + 2\hat{j} - 4\hat{k}$

$\vec{d}$ = Displacement vector = $4\hat{i} + 2\hat{j} - 2\hat{k}$

$\Rightarrow$ Work done = $\vec{F}.\vec{d}$ = (28 + 4 + 8) units = 40 units

8. Let D be the mid-point of BC. Then,

$$\overrightarrow{AD} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{2}$$

$\Rightarrow \quad \left|\overrightarrow{AD}\right| = 4\hat{i} + \hat{j} + 4\hat{k}$

$\Rightarrow \quad \left|\overrightarrow{AD}\right| = \sqrt{16 + 1 + 16} = \sqrt{33}$

Hence, required length = $\sqrt{33}$ units.

9. We have,

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|=\vec{0} \qquad \Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2=0$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$\Rightarrow \quad 1+4+9+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$\Rightarrow \quad \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=-7$

11. We have,

$$(\vec{u}+\vec{v}-\vec{w}).(\vec{u}-\vec{v})\times(\vec{v}-\vec{w})$$
$$=(\vec{u}+\vec{v}-\vec{w}).(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}-\vec{v}\times\vec{v}+\vec{v}\times\vec{w})$$
$$=(\vec{u}+\vec{v}-\vec{w}).(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}+\vec{v}\times\vec{w})$$
$$=\vec{u}.(\vec{u}\times\vec{v})-\vec{u}.(\vec{u}\times\vec{w})\times\vec{u}+(\vec{v}\times\vec{w})$$
$$+\vec{v}.(\vec{u}\times\vec{v})-\vec{v}.(\vec{u}\times\vec{w})+\vec{v}.(\vec{v}\times\vec{w})$$
$$-\vec{w}.(\vec{u}\times\vec{v})+\vec{w}.(\vec{u}\times\vec{w})-\vec{w}.(\vec{v}\times\vec{w})$$
$$=\vec{u}.(\vec{v}\times\vec{w})-\vec{v}.(\vec{u}\times\vec{w})-\vec{w}.(\vec{u}\times\vec{v})$$
$$=[\vec{u}\,\vec{v}\,\vec{w}]-[\vec{v}\,\vec{u}\,\vec{w}]-[\vec{w}\,\vec{u}\,\vec{v}]$$
$$=[\vec{u}\,\vec{v}\,\vec{w}]+[\vec{u}\,\vec{v}\,\vec{w}]-[\vec{u}\,\vec{v}\,\vec{w}]$$
$$=[\vec{u}\,\vec{v}\,\vec{w}]=\vec{u}.(\vec{v}\times\vec{w})$$

12. It is given that

$\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$

$\Rightarrow \quad \vec{a}+2\vec{b}=\lambda\vec{c}$ and $\vec{b}+3\vec{c}=\mu\vec{a}$ for some scalar λ and μ.

$\Rightarrow \quad \vec{b}+3\vec{c}=\mu(\lambda\vec{c}-2\vec{b})$

$\Rightarrow \quad (2\mu+1)\vec{b}+(3-\mu\lambda)\vec{c}=\vec{0}$

$\Rightarrow \quad 2\mu+1=0$ and $3-\mu\lambda=0$

$\Rightarrow \quad \mu=-\frac{1}{2}, \lambda=-6 \quad \left[\begin{array}{c}\because \vec{b} \text{ and } \vec{c} \\ \text{are non-collinear}\end{array}\right]$

$\therefore \quad \vec{a}+2\vec{b}=\lambda\vec{c}$

$\Rightarrow \quad \vec{a}+2\vec{b}=-6\vec{c} \Rightarrow \vec{a}+2\vec{b}+6\vec{c}=\vec{0}$

14. Let $\vec{\alpha}=\vec{a}+2\vec{b}+3\vec{c}, \vec{\beta}=\lambda\vec{b}+4\vec{c}$ and $\vec{\gamma}=(2\lambda-1)\vec{c}$.

Then, $[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=\begin{vmatrix}1 & 2 & 3\\0 & \lambda & 4\\0 & 0 & (2\lambda-1)\end{vmatrix}[\vec{a}\,\vec{b}\,\vec{c}]$

$\Rightarrow \quad [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=\lambda(2\lambda-1)\,[\vec{a}\,\vec{b}\,\vec{c}]$

$\Rightarrow \quad [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=0$, if $\lambda=0, \frac{1}{2}$ $[\because [\vec{a}\,\vec{b}\,\vec{c}]\neq 0]$

Hence, $\vec{\alpha},\vec{\beta},\vec{\gamma}$ are non-coplanar for all values of λ except two values 0 and $\frac{1}{2}$.

16. (a b) c = $1/3\,|b|\,|c|\,a$

$\Rightarrow \quad (a.c)b-(b.c)a=1/3\,|b|\,|c|\,a$

$\Rightarrow \quad (a.c)b=\left\{(b.c)+\frac{1}{3}|b||c|\right\}a$

$\Rightarrow \quad (a.c)b=|b|\,|c|\left\{\cos\theta+\frac{1}{3}\right\}a$

As a and b are not parallel, a.c=0 and $\cos\theta+\frac{1}{3}=0$

$\Rightarrow \quad \cos\theta=-\frac{1}{3}$. Hence $\sin\theta=\frac{2\sqrt{2}}{3}$

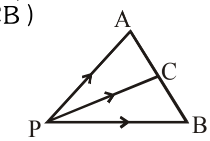
17. $\overrightarrow{PA}+\overrightarrow{PB}=(\overrightarrow{PA}+\overrightarrow{AC})+(\overrightarrow{PB}+\overrightarrow{BC})-(\overrightarrow{AC}+\overrightarrow{BC})$

$=\overrightarrow{PC}+\overrightarrow{PC}-(\overrightarrow{AC}-\overrightarrow{CB})$

$=2\overrightarrow{PC}-0$

$(\because \overrightarrow{AC}=\overrightarrow{CB})$

$\therefore \quad \overrightarrow{PA}+\overrightarrow{PB}=2\overrightarrow{PC}$



21. $[abc]=\begin{vmatrix}1 & 0 & -1\\x & 1 & 1-x\\y & x & 1+x-y\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\x & 1 & 1\\y & x & 1+x\end{vmatrix}=\begin{vmatrix}1 & 1\\x & 1+x\end{vmatrix}=1$

22. $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$

$\Rightarrow (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$\Rightarrow (\vec{b}.\vec{c})\vec{a}=(\vec{a}.\vec{b})\vec{c}$

So that $\vec{a}$ is parallel to $\vec{c}$

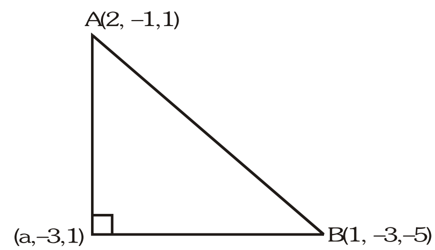
24. AC $\perp$ BC

$\therefore$ dr's of AC and BC will be (2–a,2,0) and (1–a,0,–6)

So that (2 – a) (1 – a) + 2 0 + 0 (–6) = 0

$\Rightarrow \quad a^2-3a+2=0$

$\therefore \quad a=1, 2$



29. $[3\vec{u}\;\; p\vec{v}\;\; p\vec{w}]-[p\vec{v}\;\;\vec{w}\;\; q\vec{u}]-[2\vec{w}\;\; q\vec{v}\;\; q\vec{u}]=0$

$3p^2[\vec{u}\,\vec{v}\,\vec{w}]-pq[\vec{v}\;\;\vec{w}\;\;\vec{u}]-2q^2[\vec{w}\;\;\vec{v}\;\;\vec{u}]=0$

$(3p^2-pq+2q^2).[\vec{u}\;\;\vec{v}\;\;\vec{w}]=0$

$3p^2-pq+2q^2=0$

has exactly one solution

$p=q=0$

30. $(\vec{a}\times\vec{b})+\vec{c}=0$

$(\vec{a}\times\vec{b})=-\vec{c}$

$\Rightarrow \vec{a}\times(\vec{a}\times\vec{b})=-\vec{a}\times\vec{c}$

$\Rightarrow (\vec{a}.\vec{b})\vec{a}-|\vec{a}|^2\,\vec{b}=-\vec{a}\times\vec{c}$

$\Rightarrow 3(\vec{j}-\vec{k})-2\vec{b}=-(-2i-j-k)$

$(\vec{a}\times\vec{c}=-2i-j-k)$

$\Rightarrow 2\vec{b}=(-2i+2j-4k)$

$\Rightarrow \vec{b}=-i+j-2k$

31. Give $\vec{a} \perp \vec{b}$, $\vec{a} \perp \vec{c}$ & $\vec{b} \perp \vec{c}$

so $\vec{a}.\vec{c} = 0$ & $\vec{b}.\vec{c} = 0$

$\Rightarrow \lambda - 1 + 2\mu = 0$ & $2\lambda + 4 + \mu = 0$

$\Rightarrow \lambda = -3$ & $\mu = 2$

32. a.b. $\neq$ 0

a.d = 0

b c = b d

a (b c) = a (b d)

(a.c)b – (a.b)c – (a.c)b – (a.b)d {a.d=0}

$\Rightarrow$ (a.b)d = (a.b)c (a.c)b (divide by a.b)

$$\boxed{\mathbf{d} = \mathbf{c} - \frac{(\mathbf{a.c})}{(\mathbf{a.b})}\mathbf{b}}$$

33. $\vec{a}.\vec{b} = 0$ and $|a| = |b| = 1$

$(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b}) = (\vec{a}\times\vec{b})\times\vec{a} + (\vec{a}\times\vec{b})\times 2\vec{b}$

$= -\left[\vec{a}\times(\vec{a}\times\vec{b}) + 2\vec{b}\times(\vec{a}\times\vec{b})\right]$

$= -\left[(\vec{a}.\vec{b})\vec{a} - (\vec{a}\,.\,\vec{a})\vec{b} + 2(\vec{b}\,.\vec{b})\vec{a} - 2(\vec{b}\,.\,\vec{a})\vec{b}\right]$

$= -\left[0 - \vec{b} + 2\vec{a} + 0\right] = \left[\vec{b} - 2\vec{a}\right]$

$\therefore\ (2\vec{a} - \vec{b})\,.\left[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})\right]$

$= (2\vec{a} - \vec{b})\,.\,(\vec{b} - 2\vec{a})$

$= -4a^2 - b^2 + 4\,\vec{a}\,.\,\vec{b} = -5$

34. $\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$

$p(qr - 1) - (r - 1) + (1 - q) = 0$

$pqr - p - r + 1 + 1 - q = 0$

$pqr - (p + r + q) + 2 = 0$

$pqr - (p + r + q) = -2$

35. Let

$a + 3b = \lambda\,\vec{c}$

add $6\,\vec{c}$ both side

$\vec{a} + 3\vec{b} + 6\vec{c} = (\lambda + 6)\vec{c}$

Let

$\vec{b} + 2\vec{c} = \mu\vec{a}$

$3b + 6\,\vec{c} = 3\mu\,\vec{a}$

add $\vec{a}$ both side

$\vec{a} + 3\vec{b} + 6\vec{c} = (3\mu + 1)\vec{a}$

Hence $(\lambda + 6)\vec{c} = (3\mu + 1)\vec{a}$

But given $\vec{a}$ and $\vec{c}$ are non coliner

Hence $\lambda + 6 = 3\mu + 1 = 0$

so $\vec{a} + 3\vec{b} + 6\vec{c} = \vec{0}$

36. $\vec{c}.\vec{d} = 0$

$\Rightarrow (\tilde{a} + 2\tilde{b}).(5\tilde{a} - 4\tilde{b}) = 0$

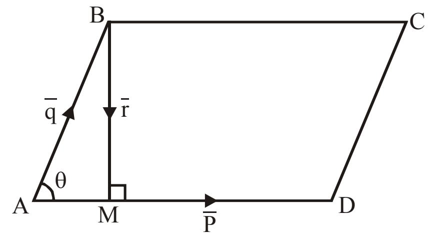
$\Rightarrow 5 - 8 + 6\,\tilde{a}\,.\,\tilde{b} = 0$

$\Rightarrow \tilde{a}\,.\,\tilde{b} = 1/2$

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = \dfrac{\pi}{3}$

37.



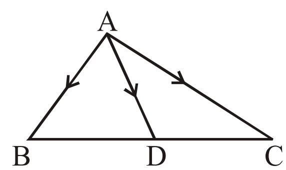
$\overline{q} + \overline{r} = \overline{AM}$

$\Rightarrow \overline{r} = -\overline{q} + \overline{AM}$

$\Rightarrow \overline{r} = -\overline{q} + \dfrac{\overline{p}.\overline{q}}{|\overline{p}|^2}\vec{p}$

$\Rightarrow \overline{r} = -\overline{q} + \left(\dfrac{\overline{p}.\overline{q}}{\overline{p}.\overline{p}}\right)\overline{p}$

38.



$\overrightarrow{AD} = \dfrac{\overrightarrow{AB} + \overrightarrow{AC}}{2} = 4\tilde{j} - \tilde{j} + 4\tilde{k}$

$|\,\overrightarrow{AD}\,| = \sqrt{33}$

1. (b) Given that $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are vectors such that

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$ ... (1)

P_1 is the plane determined by vectors $\vec{a}$ and $\vec{b}$

$\therefore$ Normal vectors $\vec{n}_1$ to P_1 will be given by

$\vec{n}_1 = \vec{a} \times \vec{b}$

Similarly P_2 is the plane determined by vectors $\vec{c}$ and $\vec{d}$

$\therefore$ Normal vector $\vec{n}_2$ to P_2 will be given by

$\vec{n}_2 = \vec{c} \times \vec{d}$

Substituting the values of $\vec{n}_1$ and $\vec{n}_2$ in equation (1) we get $\vec{n}_1 \quad \vec{n}_2 = 0$

$\Rightarrow \vec{n}_1 || \vec{n}_2$

and hence the planes will also be parallel to each other.

Thus angle between the planes = 0.

3. (a) $\tilde{a}, \tilde{b}, \tilde{c}$ are unit vectors.

$\therefore \tilde{a} \cdot \tilde{a} = \tilde{b} \cdot \tilde{b} = \tilde{c} \cdot \tilde{c} = 1$

Now, $x = |\tilde{a} - \tilde{b}|^2 + |\tilde{b} - \tilde{c}|^2 + |\tilde{c} - \tilde{a}|^2$

$= \tilde{a} \cdot \tilde{a} + \tilde{b} \cdot \tilde{b} - 2\tilde{a} \cdot \tilde{b} + \tilde{b} \cdot \tilde{b} + \tilde{c} \cdot \tilde{c} -$
$2\tilde{b} \cdot \tilde{c} + \tilde{c} \cdot \tilde{c} + \tilde{a} \cdot \tilde{a} - 2\tilde{c} \cdot \tilde{a}$

$= 6 - 2(\tilde{a}.\tilde{b} + \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a})$ ......(1)

Also $|\tilde{a} + \tilde{b} + \tilde{c}| \geq 0$

$\Rightarrow \tilde{a}.\tilde{a} + \tilde{b}.\tilde{b} + \tilde{c}.\tilde{c} + 2(\tilde{a}.\tilde{b} + \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a}) \geq 0$

$\Rightarrow 3 + 2(\tilde{a}.\tilde{b} + \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a}) \geq 0$

$\Rightarrow -2(\tilde{a}.\tilde{b} + \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a}) \leq 3$

$\Rightarrow 6 - 2(\tilde{a}.\tilde{b} + \tilde{b}.\tilde{c} + \tilde{c}.\tilde{a}) \leq 9$ ... (2)

From (1) and (2), $x \leq 9$

$\therefore$ x does not exceed 9

5. Given data is insufficient to uniquely determine the three vectors as there are only 6 equations involving 9 variables.

$\therefore$ We can obtain infinitely many set of three vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$, satisfying these conditions.

From the given data, we get

$\vec{v}_1.\vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$

$\vec{v}_2.\vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$

$\vec{v}_3.\vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$

Also $\vec{v}_1.\vec{v}_2 = -2$

$\Rightarrow |\vec{v}_1|\,|\vec{v}_2| \cos\theta = -2$

[where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$]

$\Rightarrow \cos\theta = \dfrac{-1}{\sqrt{2}} \Rightarrow \theta = 135$

Now since any two vectors are always coplanar, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in x–y plane. Let $\vec{v}_1$ is along the positive direction of x-axis then $\vec{v}_1 = 2\tilde{i}$. $[\because |\vec{v}_1| = 2]$

As $\vec{v}_2$ makes an angle 135 with $\vec{v}_1$ and lies in x–y plane, also keeping in mind $|\vec{v}_2| = \sqrt{2}$ we obtain

$\vec{v}_2 = -\tilde{i} \pm \tilde{j}$

Again let, $\vec{v}_3 = \alpha\tilde{i} + \beta\tilde{i} + \gamma\tilde{k}$

$\because \vec{v}_3.\vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \Rightarrow \alpha = 3$

and $\vec{v}_3.\vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5 \Rightarrow \beta = \pm 2$

Also $|\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29$

$\Rightarrow \gamma = \pm 4$

Hence $\vec{v}_3 = 3\tilde{i} \pm 2\tilde{j} \pm 4\tilde{k}$

Thus, $\vec{v}_1 = 2\tilde{i}; \vec{v}_2 = -\tilde{i} \pm \tilde{j}; \vec{v}_3 = 3\tilde{i} \pm 2\tilde{j} \pm 4\tilde{k}$

are some possible answers.

6. $\vec{A}$ (t) is parallel to $\vec{B}$ (t) for some $t \in [0,1]$

if and only if $\dfrac{f_1(t)}{g_1(t)} = \dfrac{f_2(t)}{g_2(t)}$ for some $t \in [0,1]$

or $f_1(t).g_2(t) = f_2(t).g_1(t)$ for some $t \in [0,1]$

Let $h(t) = f_1(t).g_2(t) - f_2(t).g_1(t)$

$h(0) = f_1(0).g_2(0) - f_2(0).g_1(0)$
$= 2 \quad 2 - 3 \quad 3 = -5 < 0$

$h(1) = f_1(1).g_2(1) - f_2(1).g_1(1)$
$= 6 \quad 6 - 2 \quad 2 = 32 > 0$

Since h is a continuous function, and $h(0).h(1) < 0$

$\Rightarrow$ there is some $t \in [0,1]$ for which $h(t)=0$

i.e., $\vec{A}$ (t) and $\vec{B}$ (t) are parallel vectors for this t.

8. Given that, $\vec{a} = a_1\tilde{i} + a_2\tilde{j} + a_3\tilde{k}$

$\vec{b} = b_1\tilde{i} + b_2\tilde{j} + b_3\tilde{k}$

$\vec{c} = c_1\tilde{i} + c_2\tilde{j} + c_3\tilde{k}$

where a_r, b_r, c_r, r = 1,2,3 are all non negative real numbers.

Also $\sum_{r=1}^{3} (a_r + b_r + c_r) = 3L$

To prove $V \le L^3$ Where V is vol. of parallelopiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$

$\therefore$ We have $V = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Rightarrow V = (a_1b_2c_3 + a_2b_3c_1+a_3b_1c_2)-(a_1b_3c_2 +a_2b_1c_3 + a_3b_2c_1)$....(1)

Now we know that AM $\ge$ GM

$\therefore \dfrac{(a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)}{3}$

$\ge [(a_1 + b_1 + c_1)\ (a_2 + b_2 + c_2)\ (a_3 + b_3 + c_3)]^{1/3}$

$\Rightarrow \dfrac{3L}{3} \ge [(a_1+b_1+c_1)\ (a_2+b_2+c_2)\ (a_3+b_3+c_3)]^{1/3}$

$\Rightarrow L^3 \ge (a_1+b_1+c_1)\ (a_2 + b_2 + c_2)\ (a_3 +b_3 + c_3)$

$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$ + 24 more such terms

$\ge a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$

[$\because$ a_r, b_r, $c_r \ge 0$ or r = 1,2,3]

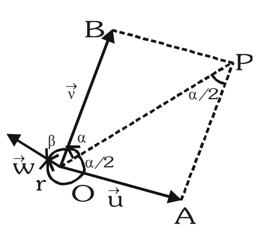
$\ge (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)$

$- (a_1b_3c_2+a_2b_1c_3 + a_3b_2c_1)$ [same reason]

= V from (1)

Thus, $L^3 \ge V$ Hence Proved

10. Given that u, v, ω are three non coplanar unit vectors. Angle between $\vec{u}$ and $\vec{v}$ is α, between $\vec{v}$ and $\vec{\omega}$ is β and between $\vec{\omega}$ and $\vec{u}$ it is γ. In fig. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent $\vec{u}$ and $\vec{v}$. Let P be a pt. on angle bisector of $\angle$AOB such that OAPB is a parallelogram.



Also $\angle$ POA = $\angle$BOP = $\alpha/2$

$\therefore \angle$ APO = $\angle$BOP = $\alpha/2$ (Alternate angles)

$\therefore$ In Δ OAP, OA = AP

$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{u} + \vec{v}$

$\therefore \widehat{OP} = \dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$ i.e. $\vec{x} = \dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

But $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}).(\vec{u} + \vec{v})$

$= 1 + 1 + 2\,\vec{u}.\vec{v}$

[$\because |\vec{u}| = |\vec{v}| = 1$]

= 2 + 2 cos $\alpha = 4\cos^2\alpha/2$.

$\therefore |\vec{u} + \vec{v}| = 2\cos\alpha/2$

$\Rightarrow \vec{x} = \dfrac{1}{2}\sec(\alpha/2)\ (\vec{u} + \vec{v})$

Similarly, $\vec{y} = \dfrac{1}{2}\sec(\beta/2)\ (\vec{v} + \vec{\omega})$

$\vec{z} = \dfrac{1}{2}\sec(\gamma/2)(\vec{\omega} + \vec{u})$

Now consider $[\vec{x} \times \vec{y}\ \ \vec{y} \times \vec{z}\ \ \vec{z} \times \vec{x}]$

$= (\vec{x} \times \vec{y}).[(\vec{y} \times \vec{z})\times(\vec{z} \times \vec{x})]$

$= (\vec{x} \times \vec{y}).[\{(\vec{y} \times \vec{z}).\vec{x}\}\vec{z} - \{(\vec{y} \times \vec{z}).\vec{z}\}\vec{x}]$

[Using def^n of vector triple product,]

$= (\vec{x} \times \vec{y}).[[\vec{x}\,\vec{y}\,\vec{z}]\vec{z} - 0]$

$= [\vec{x}\,\vec{y}\,\vec{z}]\ [\vec{x}\,\vec{y}\,\vec{z}]$ [$\because$ $[\vec{y}\,\vec{z}\,\vec{z}] = 0$]

$= [\vec{x}\,\vec{y}\,\vec{z}]^2$ ...(i)

Also $[\vec{x}\,\vec{y}\,\vec{z}] = \left[\dfrac{1}{2}\sec\dfrac{\alpha}{2}(\vec{u} + \vec{v})\ \ \dfrac{1}{2}\sec\dfrac{\beta}{2}\right.$

$\left.(\vec{v} + \vec{\omega})\ \ \dfrac{1}{2}\sec(\gamma/2)(\vec{w} + \vec{u}))\right]$

$= \dfrac{1}{8}\sec(\alpha/2)\sec(\beta/2)\sec(\gamma/2)[\vec{u} + \vec{v}\ \ \vec{v} + \vec{\omega}\ \ \vec{\omega} + \vec{u}]$

$= \dfrac{1}{8}\sec(\alpha/2)\ \sec(\beta/2)\ \sec(\gamma/2)$

$[(\vec{u} + \vec{v}).(\vec{v} + \vec{\omega}) \times (\vec{\omega} + \vec{u})]$

$= \dfrac{1}{8}\sec(\alpha/2)\ \sec(\beta/2)\ \sec(\gamma/2)$

$[(\vec{u} + \vec{v}).(\vec{v} \times \vec{\omega} + \vec{v} \times \vec{u} + \vec{\omega} \times \vec{u})]$

$= \dfrac{1}{8}\sec(\alpha/2)\ \sec(\beta/2)\ \sec(\gamma/2)[\vec{u}.\vec{v} \times \vec{\omega} + \vec{v}.\vec{\omega} \times \vec{u}]$

($\because$ $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ when ever any two vectors are same)

$= \dfrac{1}{8}\sec(\alpha/2)\ \sec(\beta/2)\ \sec(\gamma/2)\ 2[\vec{u}\,\vec{v}\,\vec{\omega}]$

$= \frac{1}{4} \sec(\alpha/2)\ \sec(\beta/2)\ \sec(\gamma/2)\ 2[\vec{u}\,\vec{v}\,\vec{\omega}]$

$\therefore\ [\vec{x}\,\vec{y}\,\vec{z}]^2 = \frac{1}{16}\ [\vec{u}\,\vec{v}\,\vec{\omega}]^2 \sec^2\alpha/2\ \sec^2\beta/2\ \sec^2\gamma/2$

....(ii)

From (i) and (ii),

$[\vec{x}\times\vec{y}\quad \vec{y}\times\vec{z}\quad \vec{z}\times\vec{x}]$

$= \frac{1}{16}\ [\vec{u}\,\vec{v}\,\vec{\omega}]^2\ \sec^2\alpha/2\ \sec^2\beta/2\ \sec^2\gamma/2.$

12. Given that $\vec{a} \neq \vec{b} \neq \vec{c} \neq \vec{d}$

Such that $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$ ...(i)

$\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ ...(ii)

To prove that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) \neq 0$

Subtracting equation (ii) from (i) we get

$\vec{a}\times(\vec{c}-\vec{b}) = (\vec{b}-\vec{c})\times\vec{d}$

$\Rightarrow\ \vec{a}\times(\vec{c}-\vec{b}) = \vec{d}\times(\vec{c}-\vec{b})$

$\Rightarrow\ \vec{a}\times(\vec{c}-\vec{b}) - \vec{d}\times(\vec{c}-\vec{b}) = 0$

$\Rightarrow\ (\vec{a}-\vec{d})\times(\vec{c}-\vec{b}) = 0 \ \Rightarrow\ \vec{a}-\vec{d}\ ||\ \vec{c}-\vec{b}$

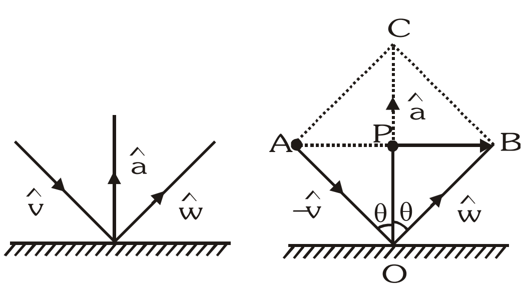
$[\because\ \vec{a}-\vec{d} \neq 0,\ \vec{c}-\vec{b} \neq 0$ as all distinct$]$

$\Rightarrow$ Angle between $\vec{a}-\vec{d}$ and $\vec{c}-\vec{b}$ is either 0 or 180 .

$\Rightarrow\ (\vec{a}-\vec{d})\,.\,(\vec{c}-\vec{b}) = |\vec{a}-\vec{d}|\,|\vec{c}-\vec{b}|$

[cos 0 or cos 180] $\neq 0$ as $\vec{a},\vec{d},\vec{c},\vec{b}$ all are different.

14. Given that incident ray is along $\tilde{v}$, reflected ray is along $\tilde{w}$ and normal is along $\tilde{a}$, outwards. The given figure can be redrawn as shown.



We know that incident ray, reflected ray and normal lie in a plane, and angle of incidence = angle of reflection.

Therefore $\tilde{a}$ will be along the angle bisector of $\tilde{w}$ and $-\tilde{v}$, i.e.,

$\tilde{a} = \frac{\tilde{w} + (-\tilde{v})}{|\tilde{w}-\tilde{v}|}$ ...(i)

[$\because$ angle bisector will along a vector dividing in same ratio as the ratio of the sides forming that angle.]

But $\tilde{a}$ is a unit vector

where $|\tilde{w}-\tilde{v}|$ = OC = 2OP

$= 2\ |\tilde{w}|\ \cos\theta = 2\cos\theta$

Substituting this value in equation (i) we get

$\tilde{a} = \frac{\tilde{w}-\tilde{v}}{2\cos\theta}$

$\therefore\ \tilde{w} = \tilde{v} + (2\cos\theta)\,\tilde{a}$

$= \tilde{v} - 2(\tilde{a}\,.\,\tilde{v})\,\tilde{a}\quad [\because\ \tilde{a}\,.\,\tilde{v} = -\cos\theta].$

15. (b) Normal to plane P_1 is

$\tilde{n}_1 = (2\tilde{j}+3\tilde{k})\times(4\tilde{j}-3\tilde{k}) = -18\tilde{i}$

Normal to plane P_2 is

$\tilde{n}_2 = (\tilde{j}-\tilde{k})\times(3\tilde{i}+3\tilde{j}) = 3\tilde{i}-3\tilde{j}-3\tilde{k}$

$\therefore\ \vec{A}$ is parallel to $\pm(\tilde{n}_1\times\tilde{n}_2) = \pm(-54\tilde{j}+54\tilde{k})$

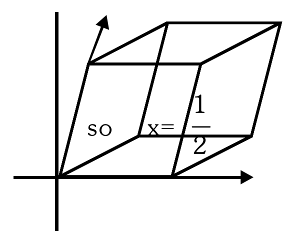
Now, angle between $\vec{A}$ and $2\tilde{i}+\tilde{j}-2\tilde{k}$ is given

by $\cos\theta = \pm\frac{(-54\tilde{j}+54\tilde{k}).(2\tilde{i}+\tilde{j}-2\tilde{k})}{54\sqrt{2}.3} = \pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

19. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{a} = \hat{i}$ then $\vec{b} = \frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}$



& $\frac{x}{2}+\frac{y\sqrt{3}}{2} = \frac{1}{2}$

$\Rightarrow\ y\sqrt{3} = \frac{1}{2}\quad \therefore\ y = 1/2\sqrt{3}$

also $x^2+y^2+z^2 = 1$

$\Rightarrow\ z^2 = 2/3 \Rightarrow z = \pm\sqrt{2/3}$

so volume $= \begin{vmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 1/2 & 1/2\sqrt{3} & \pm\sqrt{2/3} \end{vmatrix} = 1/\sqrt{2}$

Alternative

volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

$\sqrt{\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}} = \sqrt{\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}} = \frac{1}{\sqrt{2}}$

20. $\left|\overrightarrow{OP}\right| = \left|\tilde{a}\cos t + \tilde{b}\sin t\right|$

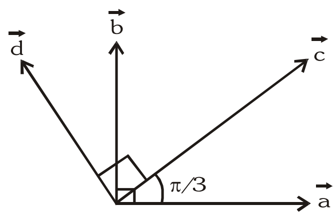
$= (\cos^2 t + \sin^2 t + 2\sin t\cos t\ \tilde{a}\,.\,\tilde{b})^{1/2}$

$= (1 + \sin 2t\,\tilde{a}\,\tilde{b})^{1/2}$

$\therefore\ \left|\overrightarrow{OP}\right|_{max} = (1+\tilde{a}\,\tilde{b})^{1/2}$,when $t = \dfrac{\pi}{4}$

Now $\tilde{u} = \dfrac{\tilde{a}+\tilde{b}}{\sqrt{2}\dfrac{|\tilde{a}+\tilde{b}|}{\sqrt{2}}} = \dfrac{\tilde{a}+\tilde{b}}{|\tilde{a}+\tilde{b}|}$

21. $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 1$ ...(1)



Let $\vec{a}\wedge\vec{b} = \alpha$

$\vec{a}\wedge\vec{b} = \beta$

angle between plane of $(\vec{a},\vec{b})$ & $(\vec{c},\vec{d})$ be θ

equation (1) becomes

$\sin\alpha\,.\,\sin\beta\cos\theta = 1$

$\Rightarrow \alpha = \dfrac{\pi}{2},\ \beta = \dfrac{\pi}{2},\ \theta = 0$

$\Rightarrow \vec{b}$ & $\vec{d}$ are non-parallel.

22. (A) $2\sin^2\theta + \sin^2 2\theta = 2$

$\sin^2\theta + 2\sin^2\theta\cos^2\theta = 1$

$t + 2t(1-t) = 1$

$t + 2t - 2t^2 = 1$

$2t^2 - 3t + 1 = 0$

$(2t-1)(t-1) = 0$

$t = 1,\ 1/2$

$\sin^2\theta = 1,\ 1/2$

(B) $\dfrac{6x}{\pi} = I_1$ & $\dfrac{3x}{\pi} = I_2$

$\Rightarrow\ x = \dfrac{I_1\pi}{6} = \dfrac{\pi}{6},\dfrac{\pi}{3},\dfrac{\pi}{2},\pi$.

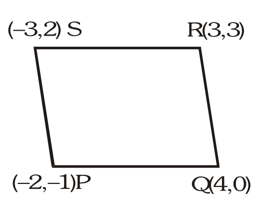
(C) $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

(D) $\vec{a}+\vec{b}+\sqrt{3}\vec{c} = 0$

$\Rightarrow a^2+b^2+2\vec{a}.\vec{b} = 3c^2 \Rightarrow 2 + 2\cos\theta = 3$

$\Rightarrow\ \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$.

23. Ans. (A)



$\overrightarrow{PQ} = 6\tilde{i}+\tilde{j}$

$\overrightarrow{SR} = 6\tilde{i}+\tilde{j}$

$\therefore\ \overrightarrow{PQ} = \overrightarrow{SR}$

$\overrightarrow{PS} = -\tilde{i}+3\tilde{j}$

$\overrightarrow{QR} = -\tilde{i}+3\tilde{j}$

$\therefore\ \overrightarrow{PS} = \overrightarrow{QR}$

But $\overrightarrow{PQ}.\overrightarrow{PS} = -6+3 = -3 \neq 0$ & $\left|\overrightarrow{PQ}\right| \neq \left|\overrightarrow{PS}\right|$

$\Rightarrow$ PQRS is a parallelogram but neither a rhombus nor a rectangle.

24. $(2\vec{a}+\vec{b})\,.\,[(\vec{a}\times\vec{b})\times\vec{a} - 2(\vec{a}\times\vec{b})\times\vec{b}]$

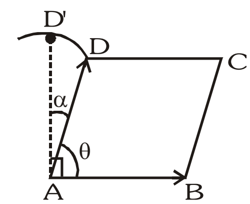
$= (2\vec{a}+\vec{b})\,.\,[a^2\vec{b} - (\vec{a}\,.\,\vec{b})\vec{a} - 2\{(\vec{a}\,.\,\vec{b})\vec{b} - b^2\vec{a}\}]$

$= (2\vec{a}+\vec{b})\,.\,[a^2\vec{b} + 2b^2\vec{a}]$; as $\vec{a}\,.\,\vec{b} = 0$

$= (2\vec{a}+\vec{b})\,.\,[2\vec{a}+\vec{b}]$ as $[a^2 = b^2 = 1]$

$\Rightarrow 4a^2 + b^2 = 5$

25. TLet θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$



$\Rightarrow\ \theta + \alpha = 90$

$\Rightarrow\ \alpha = 90 - \theta$

$\Rightarrow\ \cos\alpha = \sin\theta$ ....(i)

Now, $\cos\theta = \dfrac{\overrightarrow{AB}.\overrightarrow{AD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|} = \dfrac{8}{9}$

$\Rightarrow\ \cos\theta = \dfrac{\sqrt{17}}{9}$ from (i).

26. **(a)** $\vec{v} = x\vec{a}+y\vec{b}$

$= \tilde{i}(x+y)+\tilde{j}(x-y)+\tilde{k}(x+y)$ ....(i)

Given, $\vec{v}.\tilde{c} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow\ \dfrac{x+y-x+y-x-y}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$y - x = 1$

$\Rightarrow\ x - y = -1$ ....(ii)

using (ii) in (i) we get

$\vec{v} = (x+y)\tilde{i} - \tilde{j} + (x+y)\tilde{k}$

(b) $\vec{a} = \tilde{i}+\tilde{j}+2\tilde{k}$

$\vec{b} = \tilde{i}+2\tilde{j}+\tilde{k}$

$\vec{c} = \tilde{i}+\tilde{j}+\tilde{k}$

$\vec{v} = \lambda((\vec{a}\times\vec{b})\times\vec{c}) = \lambda((\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$\vec{v} = \lambda[4(\tilde{i}+2\tilde{j}+\tilde{k}) - 4(\tilde{i}+\tilde{j}+2\tilde{k})]$

$\vec{v} = 4\lambda(\tilde{j}-\tilde{k})$

(c) $\vec{a} = -\hat{i} - \hat{k}$

$\vec{b} = -\hat{i} + \hat{j}$

$\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

Taking cross product by $\vec{a}$

$(\vec{r} \times \vec{b}) \times \vec{a} = (\vec{c} \times \vec{b}) \times \vec{a}$

$\Rightarrow (\vec{r}.\vec{a})\vec{b} - (\vec{b}.\vec{a})\vec{r} = (\vec{c}.\vec{a})\vec{b}) - (\vec{b}.\vec{a})\vec{c}$

$\Rightarrow 0 - \vec{r} = (-1-3)(-\hat{i} + \hat{j}) - (1)(\hat{i} + 2\hat{j} + 3\hat{k})$

$\vec{r} = -3\hat{i} + 6\hat{j} + 3\hat{k}$

$\vec{r}.\vec{b} = 3 + 6 = 9$

27. (a) $\left|\vec{a} - \vec{b}\right|^2 + \left|\vec{b} - \vec{c}\right|^2 + \left|\vec{c} - \vec{a}\right|^2 = 9$

$\Rightarrow \quad 6 - 2\Sigma\vec{a}.\vec{b} = 9$

$\Rightarrow \quad \Sigma\vec{a}.\vec{b} = -\dfrac{3}{2} \qquad \ldots(1)$

$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 \geq 0$

$\Sigma\vec{a}^2 + 2\Sigma\vec{a}.\vec{b} \geq 0$

$\Sigma\vec{a}.\vec{b} \geq -\dfrac{3}{2}$

for equality $|\vec{a} + \vec{b} + \vec{c}| = 0$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

$5\vec{b} + 5\vec{c} = -5\vec{a}$

$2\vec{a} + 5\vec{b} + 5\vec{c} = -3\vec{a}$

$\left|2\vec{a} + 5\vec{b} + 5\vec{c}\right| = 3\left|\vec{a}\right| = 3$

(b) $(\vec{a} + \vec{b}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0$

$\Rightarrow \quad \vec{a} + \vec{b} = \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$

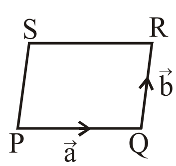
$|\vec{a} + \vec{b}| = \sqrt{29} \;\Rightarrow\; |\lambda| = 1$

$\vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} + 4\hat{k})$

$(\vec{a} + \vec{b}).(-7\hat{i} + 2\hat{j} + 3\hat{k})$

$= -14 + 6 + 12 = 4$

28. $\vec{a} + \vec{b} = \overrightarrow{PR}$ & $\vec{a} - \vec{b} = \overrightarrow{QS}$



$\vec{a} = \dfrac{\overrightarrow{PR} + \overrightarrow{QS}}{2}$ & $\vec{b} = \dfrac{\overrightarrow{PR} - \overrightarrow{QS}}{2}$

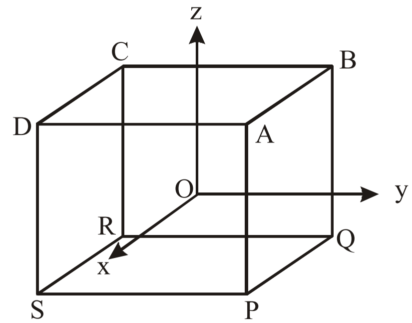
$\vec{a} = 2\hat{i} - \hat{j} - 3\hat{k}$ & $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

$$\text{Volume} = \begin{vmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

2(4) + (3 – 1) – 3(2 – 2)

8 + 2 = 10

29.



O is at the centre of cube ABCDPQRS

The 8 vectors will represent

$\overrightarrow{OA}, \overrightarrow{OB} \ldots\ldots \overrightarrow{OD}, \overrightarrow{OP}, \ldots\ldots \overrightarrow{OS}$

any three out of these 8 will be coplanar when two of them are collinear. There are 4 pairs of collinear vectors

$\overrightarrow{OA}$ & $\overrightarrow{OR}$, $\overrightarrow{OB}$ & $\overrightarrow{OS}$, $\overrightarrow{OC}$ & $\overrightarrow{OP}$, $\overrightarrow{OD}$ & $\overrightarrow{OQ}$

(it will generate 4   6 = 24 set of coplanar vectors) rest of the combination of 3 vectors will form three edges of a tetrahedron so they will be not coplanar. So number of non-coplanar vectors

${}^8C_3$ – 4.6 = 32

30. (P) Given $[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$[2(\vec{a} \times \vec{b})\ \ 3(\vec{b} \times \vec{c})\ \ (\vec{c} \times \vec{a})]$

$= 6[\vec{a} \times \vec{b}\ \ \vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}] = 6[\vec{a}\ \vec{b}\ \vec{c}]^2 = 24$

(Q) Given $[\vec{a}\ \vec{b}\ \vec{c}] = 5$

$[3(\vec{a} + \vec{b})\ \ (\vec{b} + \vec{c})\ \ 2(\vec{c} + \vec{a})]$

$= 12[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 60$

(R) Given $\dfrac{1}{2}|\vec{a} \times \vec{b}| = 20 \;\Rightarrow\; |\vec{a} \times \vec{b}| = 40$

$\left|\dfrac{1}{2}(2\vec{a} + 3\vec{b}) \times (\vec{a} - \vec{b})\right| = \dfrac{1}{2}\left|0 + 3\vec{b} \times \vec{a} - 2\vec{a} \times \vec{b}\right|$

$= \dfrac{1}{2}|-5\vec{a} \times \vec{b}| = \dfrac{5}{2}|\vec{a} \times \vec{b}| = \dfrac{5}{2}.40 = 100$

(S) Given $|\vec{a} \times \vec{b}| = 30$

$|(\vec{a} + \vec{b}) \times \vec{a}| = |0 + \vec{b} \times \vec{a}| = 30$