UNIT # 01

LOGARITHMS, QUADRATIC EQUATIONS, TRIGNOMETRIC RATIOS AND IDENTITIES

LOGARITHMS

EXERCISE - 01 — CHECK YOUR GRASP

2. Using property we get

$$\frac{a^4-(a^2+1)-2a}{a^2-a-1}=\frac{a^4-(a+1)^2}{a^2-a-1}=a^2+a+1$$

4. $\frac{\log b}{\log b+\log a+\log c}+\frac{\log c}{\log c+\log a+\log b}+\frac{\log a}{\log c+\log a+\log b}=1$

6. $(1+k)^n=\frac{s}{p}$ $\Rightarrow n\log(1+k)=\log(s/p)$

$$\Rightarrow n=\frac{\log s/p}{\log(1+k)}$$

7. $1+x>0,\ 1-x>0,\ 1-x^2>0,\quad x\neq 0$

8. $(7x-9)^2(3x-4)^2=100$

$\Rightarrow (21x^2-55x+36)^2=100$

$\Rightarrow\quad 21x^2-55x+36=\pm\ 10$

$21x^2-55x+26=0$

$$x=\frac{55\pm\sqrt{3025-2184}}{42}=\frac{55\pm 29}{42}=2,\ \frac{13}{21}$$

only two real solution

9. Let $\log_2 x=y$

$\Rightarrow\quad 1+2y+y^2+y+2y^2+y^3=1$

$\Rightarrow\quad y(y^2+3y+3)=0$

$\Rightarrow\quad y=0\quad$ or $\quad y^2+3y+3=0$

$\Rightarrow\quad \log_2 x=0\quad$ or $\quad D<0$ no real solution

$\Rightarrow\quad x=1$ (which is not in domain as x is in the base in one term)

10. Take log on both sides of equation & solve the equation simultaneously.

11. Use $a^{\log_b c}=c^{\log_b a}$

$\Rightarrow\ 3^{\log_4 5}+4^{\log_5 3}-3^{\log_4 5}-4^{\log_5 3}=0$

12. $$\frac{\log x}{\log\frac{p}{q}}=\frac{\log x}{\log p-\log q}=\frac{1}{\frac{\log p}{\log x}-\frac{\log q}{\log x}}$$

$$=\frac{1}{\log_x p-\log_x q}=\frac{1}{\frac{1}{\alpha}-\frac{1}{\beta}}=\frac{\alpha\beta}{\beta-\alpha}$$

13. $B=\frac{12}{3+\sqrt{5}+\sqrt{8}}$

$$B=\left(\frac{12(3+\sqrt{5}-\sqrt{8})}{(3+\sqrt{5})^2-8}\right)=\frac{12(3+\sqrt{5}-\sqrt{8})}{6+6\sqrt{5}}$$

$$=\left(\frac{2(3+\sqrt{5}-2\sqrt{2})}{1+\sqrt{5}}\right)=\frac{6+2\sqrt{5}}{\sqrt{5}+1}-\frac{4\sqrt{2}}{\sqrt{5}+1}$$

$$=\frac{(\sqrt{5}+1)^2}{\sqrt{5}+1}-\frac{4\sqrt{2}(\sqrt{5}-1)}{4}$$

$=\sqrt{5}+1-\sqrt{10}+\sqrt{2}=A\quad\Rightarrow\quad \log_A B=1$

14. $(\log_c 2)(\log_b 625)=(\log_{10}16)(\log_c 10)$

$\Rightarrow\quad \frac{\log_c 2}{\log_c 10}\times\log_b 625=\log_{10}16$

$\Rightarrow\quad \log_{10}2\quad \log_b 625=\log_{10}2^4$

$\Rightarrow\quad \log_{10}2\quad \log_b 625=4\log_{10}2$

$\Rightarrow\quad \log_b 625=4$

$\Rightarrow\quad b^4=625\ \Rightarrow\ b^4=5^4\ \Rightarrow\ b=5$

15. $x=\left(\frac{5}{3}\right)^{-100}\Rightarrow \log_{10}x=-100(\log 5-\log 3)$

$=-100\ \left(\log_{10}10-\log_{10}2-\log_{10}3\right)$

$=-100(1-.3010-.4771)$

$=-22.19=\overline{23}.81$ hence $\quad$ 0's $=23-1=22$

16. $\frac{\log_{12}x}{\log_2 x}(\log_2 xy)=1\ \Rightarrow\ \log_{12}2\log_2 xy=1$

$\Rightarrow\ \log_{12}xy=1\quad\Rightarrow\quad xy=12$

& $\quad \log_2 x\ .\ \log_3(x+y)\ .\ \log_x 3=3$

$\Rightarrow\quad \log_2 x\ \log_x(x+y)=3\quad\Rightarrow\quad \log_2(x+y)=3$

$\Rightarrow\quad x+y=8\quad\Rightarrow\quad (x,y)=(6,2)$ or $(2,6)$

17. $x^{3\log_{10}^3 x-\frac{2}{3}\log_{10}x}=10^{\frac{7}{3}}$

$\Rightarrow\quad \left(3\log_{10}^3 x-\frac{2}{3}\log_{10}x\right)\log_{10}x=\frac{7}{3}$

Put $\quad \log_{10}x=t$

$\Rightarrow\quad 3t^4-\frac{2}{3}t^2=\frac{7}{3}\quad\Rightarrow\quad 9t^4-2t^2-7=0$

$\Rightarrow\quad (9t^2+7)(t^2-1)=0\quad\therefore\quad t=\pm 1$

$\Rightarrow\quad \log_{10}x=\pm 1\ \Rightarrow\ x=10^{\pm 1}$

$\Rightarrow\quad x=10,\ \frac{1}{10}$

$\therefore\quad x_1.\ x_2=1,\ \log_{x_2}x_1=-1\ ,\ \log(x_1\ .\ x_2)=0$

EXERCISE - 02 — BRAIN TEASERS

4. (A) $\log_{10}5(2\log_{10}2 + \log_{10}5) + (\log_{10}2)^2$
$= (\log_{10}2 + \log_{10}5)^2 = (\log_{10}10)^2 = 1$

(B) $\dfrac{\log 4 + \log 3}{2\log 4 + \log 3 - \log 4} = 1$

(C) $-\log_5\log_3 3^{1/5} = -\log_5 \dfrac{1}{5} = 1$

(D) $\dfrac{1}{6}\log_{\sqrt{\frac{3}{4}}}\left(\dfrac{4}{3}\right)^3 = \log_{3/4}\dfrac{4}{3} = -1$

5. Let $\log_3 2 = y$

$N = \dfrac{1+2y}{(1+y)^2} + \dfrac{1}{(\log_2 6)^2}$

$= \dfrac{1+2y}{(1+y)^2} + \dfrac{1}{\left(1+\dfrac{1}{y}\right)^2} = \dfrac{1+2y+y^2}{(1+y)^2} = 1$

$\log_7 6 < 1 < \log_3 \pi$

6. (A) $\log_3 19 . \log_{1/7} 3 . \log_4 \dfrac{1}{7} = \log_3 19 \log_4 3$

$= \log_4 19 > 2$

(B) $\dfrac{1}{5} > \dfrac{1}{23} > \dfrac{1}{25}$

$\log_5 \dfrac{1}{5} > \log_5 \dfrac{1}{23} > \log_5 \dfrac{1}{25}$

(C) $m = 7$ & $n = 7^4 \Rightarrow n = m^4$

(D) $\log_{\sqrt{5}} 5^2 = 4$

7. Only value of x satisfying given equation are 1 & 4.

8. $\log_p \log_p (p)^{1/p^n} = \log_p \dfrac{1}{p^n} = \log_p p^{-n} = -n$

10. $\log_p q + \log_q r + \log_r p = 0$

then $(\log_p q)^3 + (\log_q r)^3 + (\log_r p)^3$

$= 3(\log_p q . \log_q r . \log_r p) = 1$

EXERCISE - 03 — MISCELLANEOUS TYPE QUESTIONS

Fill in the blanks :

3. $\log_{10}^2 x + 2\log_{10} x + 1 = \log_{10}^2 2$

$\Rightarrow (\log_{10}x + 1)^2 = \log_{10}^2 2$

$\Rightarrow \log_{10}x + 1 = \pm \log_{10}2$

$\Rightarrow \log_{10}10x = \pm \log_{10}2$

$\Rightarrow x = \dfrac{1}{5}, \dfrac{1}{20}$

4. $0.05^{\log_{\sqrt{20}}\frac{1}{9}} = \left(\dfrac{1}{20}\right)^{2\log_{20}9^{-1}} = 20^{\log_{20}9^2} = 9^2$

8. Given $a + b = ab$

$\dfrac{(a^3-1)(b^3-1)-1}{ab(a+b)} = \dfrac{a^3b^3 - a^3 - b^3 + 1 - 1}{ab(a+b)}$

$= \dfrac{(a+b)^3 - (a^3+b^3)}{ab(a+b)}$

$= \dfrac{(a+b)^3 - (a+b)^3 + 3ab(a+b)}{ab(a+b)} = 3$

Match the Column :

1. (A) $2\log_{10}(x-3) = \log_{10}(x^2-21)$

$\Rightarrow (x-3)^2 = x^2 - 21$

$\Rightarrow 6x = 30 \quad \Rightarrow x = 5$

(B) $x^{\log_2 x + 4} = 32$

$\Rightarrow (\log_2 x + 4)\log_2 x = \log_2 32$

Let $\log_2 x = y$

$\Rightarrow y^2 + 4y - 5 = 0$

$\Rightarrow (y+5)(y-1) = 0$

$\log_2 x = -5 \qquad \log_2 x = 1$

$\Rightarrow x = \dfrac{1}{32}$ & $x = 2$

(C) $5^{\log_{10}x} + \dfrac{5^{\log_{10}x}}{5} = 3.3^{\log_{10}x} + \dfrac{3^{\log_{10}x}}{3}$

$\Rightarrow \left(\dfrac{6}{5}\right)5^{\log_{10}x} = \left(\dfrac{10}{3}\right)3^{\log_{10}x}$

$\Rightarrow \left(\dfrac{5}{3}\right)^{\log_{10}x} = \left(\dfrac{5}{3}\right)^2$

$\Rightarrow \log_{10} x = 2 \Rightarrow x = 100$

(D) $9.9^{\log_3 x} - 3.3^{\log_3 x} - 210 = 0$

$\Rightarrow 9x^2 - 3x - 210 = 0$

$\Rightarrow 3x^2 - x - 70 = 0$

$\Rightarrow 3x^2 - 15x + 14x - 70 = 0$

$\Rightarrow x = 5 \; ; \; x = \dfrac{-14}{3}$ (Reject)

Assertion & Reason :

3. $-\log_{2+|x|}(5 + x^2) = \log_{3+x^2}(15 + \sqrt{x})$

$\because$ LHS < 0 and RHS > 0

hence no solution.

Comprehension :

2. $-\log_{20} 40 < 0$

2. (a) $\log_{1/3}\sqrt[4]{3^6.3^{-2}} = \log_{1/3} 3 = -1$

(b) $a^{\frac{\log_b(\log_b N)}{\log_b a}} = a^{\log_a(\log_b N)} = \log_b N$

4. $\frac{1}{\log_2(e-1)} + \log_2(e-1) \ge 2 \quad \left[x + \frac{1}{x} \ge 2 \text{ if } x > 0\right]$

6. $49^A = 49^{1-\log_7 2} = \frac{49}{49^{\log_7 2}} = \frac{49}{7^{\log_7 4}} = \frac{49}{4}$

$5^B = 5^{-\log_5 4} = \frac{1}{4}$

9. $\frac{\log_a x}{\log_a y} = 4 \quad \& \quad \frac{\log_a z}{\log_a y} = 7$

$\Rightarrow \frac{\log_a x}{4} = \frac{\log_a y}{1} = \frac{\log_a z}{7} = \lambda$

$\Rightarrow x = a^{4\lambda},\ y = a^{\lambda},\ z = a^{7\lambda}$

Now $\log_a a^{4\lambda} \log_a (a^{4\lambda+\lambda+7\lambda}) = 48$

$48\lambda^2 = 48 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

10. (a) $\left(\frac{81^{\log_9 5} + 3^{3\log_3\sqrt{6}}}{409}\right)\left(7^{\log_7 25} - 5^{\log_5 6^{3/2}}\right)$

$= \left(\frac{25 + 6^{3/2}}{409}\right)\left(25 - 6^{3/2}\right) = \frac{625 - 216}{409} = 1$

(b) $5^{\log_5 2} + \log_{\sqrt{2}}(\sqrt{7} - \sqrt{3}) + \log_{1/2}\frac{10 - 2\sqrt{21}}{16}$

$= 2 + \log_{\sqrt{2}}(\sqrt{7} - \sqrt{3}) + \log_{1/2}\left(\frac{\sqrt{7} - \sqrt{3}}{4}\right)^2$

$= 2 + \log_{\sqrt{2}}(\sqrt{7} - \sqrt{3})$

$+2\left[\log_{1/2}(\sqrt{7} - \sqrt{3}) - \log_{1/2} 4\right]$

$= 2 + 2\log_2(\sqrt{7} - \sqrt{3}) - 2\log_2(\sqrt{7} - \sqrt{3}) + 4$

$= 6$

(c) $4^{\frac{10}{5}\log_2(3-\sqrt{6}) - \frac{6}{3}\log_2\left(\sqrt{3}-\sqrt{2}\right)}$

$= 4^{\log_2\left(\frac{3-\sqrt{6}}{\sqrt{3}-\sqrt{2}}\right)^2} = 4^{\log_2 3} = 2^{\log_2 9} = 9$

11. (a) $5^{\log_a x} + 5.5^{\log_a x} = 3$

$\Rightarrow 5^{\log_a x} = \frac{1}{2} \Rightarrow \log_a x \ \log_a 5 = \log_a 2^{-1}$

$\Rightarrow \log_a x = \frac{\log_a 2^{-1}}{\log_a 5} = \log_5 2^{-1}$

$\Rightarrow x = a^{\log_5 2^{-1}} = 2^{-\log_5 a}$

(b) Let $\log_2 x = y$

$\Rightarrow \frac{1}{y}\cdot\frac{1}{y+1} = \frac{1}{2+y}$

$\Rightarrow y^2 + y = 2 + y \Rightarrow y = \pm\sqrt{2}$

$\Rightarrow \log_2 x = \pm\sqrt{2} \Rightarrow x = 2^{\pm\sqrt{2}}$

12. (a) $(x^2 + x - 6)^2 = (x + 1)^4$

$\Rightarrow x^2 + x - 6 = (x + 1)^2$

$\Rightarrow x = -7$ (reject)

or $x^2 + x - 6 = -(x + 1)^2$

$\Rightarrow x = 1, -\frac{5}{2}$ (reject)

(b) $x(\log_{10}5 - 1) = \log_{10}\left(\frac{1+2^x}{6}\right)$

$\Rightarrow \log_{10}\left(\frac{1}{2}\right)^x = \log_{10}\left(\frac{1+2^x}{6}\right)$

$\Rightarrow \frac{1}{2^x} = \frac{1+2^x}{6}$

$\Rightarrow x = 1$

13. $a^2 + b^2 = c^2 \quad a > 0,\ b > 0,\ c > 0$

$\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \log_{c-b} a$

LHS $= \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)}$

$= \frac{\log_a(c^2 - b^2)}{\log_a(c+b)\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b)\log_a(c-b)}$

$= \frac{2}{\log_a(c+b)\log_a(c-b)} = 2\log_{c+b} a \log_{c-b} a$

16. (a) Take log on both side

$200 \log_{10}5 = 200(1 - \log_{10}2) = 139.8$

$\Rightarrow$ number of integer is 140

(b) $15(\log_{10}3 + \log_{10}2) = 11.67$

$\Rightarrow$ number of integer is 12

(c) $-100 \log 3 = -47.71 = \overline{48}.29$

$\Rightarrow$ 47 zero

1. $x = 50\log_{10} x \Rightarrow 10^x = x^{50}$

$\Rightarrow 10^{2x} = x^{50\times 2} \Rightarrow 100^x = x^{100} \Rightarrow x = 100$

3. $a = \log_{12} 18 \Rightarrow 12^a = 18;$

$b = \log_{24} 54 \Rightarrow 24^b = 54 \Rightarrow 24^b = 3\,(12)^a$

$\Rightarrow 2^{3b}3^b = 2^{2a}3^{a+1} \Rightarrow 2^{3b}\,3^b = 3^{a+1}2^{2a}$

$\Rightarrow 3b = 2a \quad \& \quad a + 1 = b$

$\Rightarrow a = -3, \quad b = -2$

5. $$\frac{(\log a)^2}{\log b \log c} + \frac{(\log b)^2}{\log a \log c} + \frac{(\log c)^2}{\log a \log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 - 3\log a \log b \log c = 0$

$\Rightarrow \log a + \log b + \log c = 0$ (as a,b,c are distinct)

$\Rightarrow \log abc = 0 \Rightarrow abc = 1$

6. $$\log_7 10 - \frac{\log_7 13}{\log_7 11} = \frac{\log_7 10.\log_7 11 - \log_7 13}{\log_7 11}$$

$$= \frac{\log_7\left(7.\frac{10}{7}\right)\log_7\left(7.\frac{11}{7}\right) - \log_7\left(7.\frac{13}{7}\right)}{\log_7 11}$$

$$= \frac{\left(1+\log_7\frac{10}{7}\right)\left(1+\log_7\frac{11}{7}\right) - \left(1+\log_7\frac{13}{7}\right)}{\log_7 11}$$

$$= \frac{1+\log_7\frac{10}{7}+\log_7\frac{11}{7}+\log_7\frac{10}{7}\log_7\frac{11}{7} - 1 - \log_7\frac{13}{7}}{\log_7 11}$$

$$= \frac{\log_7\frac{110/49}{13/7} + \log_7\frac{10}{7}\log_7\frac{11}{7}}{\log_7 11}$$

$$= \frac{\log_7\frac{110}{91} + \log_7\frac{10}{7}\log_7\frac{11}{7}}{\log_7 11} > 0$$

7. Take log on both side & solve the equation simultaneously.

9. $\log 4 + \log 3 + \log 3^{1/2x} = \log\left(3^{1/x} + 27\right)$

$\Rightarrow 12.3^{\frac{1}{2x}} = 3^{1/x} + 27$

Let $3^{1/2x} = y \Rightarrow y^2 - 12y + 27 = 0$

$\Rightarrow y = 9$ or $y = 3 \Rightarrow x = \frac{1}{4}$ and $x = \frac{1}{2}$

but $x \in N \geq 2$ have no solution

10. $\log_{10}(2000xy) - \log_{10} x.\log_{10} y = 4$

$= \log_{10} 2000 + \log_{10} x + \log_{10} y - \log_{10} x \log_{10} y = 4$

$= 3 + \log_{10} 2 + \log_{10} x + \log_{10} y - \log_{10} x \log_{10} y = 4$

$= \log_{10} x + \log_{10} y - \log_{10} x \log_{10} y = 1 - \log_{10} 2 = \log_{10} 5$

Let $\log_{10}x = a$, $\log_{10}y = b$, $\log_{10}z = c$

$\Rightarrow a + b - ba = \log_{10} 5$ ....(i)

Similarly

$b + c - bc = 1 - \log_{10} 2 = \log_{10} 5$ ....(ii)

& $a + c - ac = 0$ ....(iii)

$\Rightarrow a = c = 0, 2 \Rightarrow b = \log_{10}5, \log_{10}20$

& $b = 1$ (does not give any solution)

11. $$\log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

$$\Rightarrow 2\log^2\left(\frac{x+4}{x}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

$$\Rightarrow \log\left(\frac{x+4}{x}\right) = \pm\log\left(\frac{3-x}{x-1}\right)$$

$\Rightarrow \frac{x+4}{x} = \frac{3-x}{x-1} \Rightarrow x = \pm\sqrt{2}$

$\Rightarrow x = \sqrt{2}$ only sol. and satisfying domain

also $\frac{x+4}{x} = \frac{x-1}{3-x} \Rightarrow x = \pm\sqrt{6}$

$\Rightarrow x = \sqrt{6} \in$ domain

13. $\log_{100}|x+y| = \frac{1}{2}$ and

$\log_{10} y - \log_{10}|x| = \log_{100} 4 = \log_{10} 2$

$\Rightarrow |x+y| = 10 \quad \& \quad \frac{y}{|x|} = 2 \Rightarrow y = 2|x|$

when $x > 0$; $y = 2x$

$\Rightarrow x = \frac{10}{3} \quad \& \quad y = \frac{20}{3}$

when $x < 0$; $y = -2x \Rightarrow |-x| = 10$

$\Rightarrow x = -10 \quad y = 20$

14. $2\log_2 \log_2 x + \log_{1/2}\left(\log_2(2\sqrt{2}x)\right) = 1$

$\Rightarrow 2\log_2\log_2 x - \log_2\log_2(2\sqrt{2}x) = 1$

$$\Rightarrow \log_2\left(\frac{(\log_2 x)^2}{\log_2 2\sqrt{2}x}\right) = 1$$

$\Rightarrow (\log_2 x)^2 = 2\log_2 2\sqrt{2}x$

$\Rightarrow (\log_2 x)^2 = \log_2 8x^2 = 3 + 2\log_2 x$

$\Rightarrow y^2 - 2y - 3 = 0$ (where $y = \log_2 x$)

$\Rightarrow y = 3 ; \quad y = -1$

$\Rightarrow x = 8 ; \quad x = 1/2$ (Not in domain)

2. Ans. (C)

$(2x)^{\ell n2} = (3y)^{\ell n3}$

$\Rightarrow$ $\ell n2\ (\ell n2 + \ell nx) = \ell n3(\ell n3 + \ell ny)$ .....(i)

$3^{\ell nx} = 2^{\ell ny}$

$\Rightarrow$ $(\ell nx)\ (\ell n3) = (\ell ny)\ (\ell n2)$ .....(ii)

using (ii) in (i)

$\Rightarrow$ $\ell n2(\ell n2+\ell nx)= \ell n3\left(\ell n3 + \frac{(\ell nx)(\ell n3)}{\ell n2}\right)$

$\Rightarrow$ $\ell n^2 2 - \ell n^2 3 = \ell nx\left\{\frac{\ell n^2 3}{\ell n2} - \ell n2\right\}$

$\Rightarrow$ $\ell nx = -\ \ell n2$

$\Rightarrow$ $x = \frac{1}{2}$

3. Let $y = \sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots\ldots}}}$

$\Rightarrow$ $y^2 = 4 - \frac{1}{3\sqrt{2}}\ y$ $\Rightarrow$ $3\sqrt{2}\ y^2 = 12\sqrt{2} - y$

$\Rightarrow$ $3\sqrt{2}y^2 + y - 12\sqrt{2} = 0$

$\Rightarrow$ $(3y - 4\sqrt{2})\ (\sqrt{2}y + 3) = 0$

$\Rightarrow$ $y = \frac{4\sqrt{2}}{3}$; $y = -\frac{3}{\sqrt{2}}$ (reject)

$\therefore$ $V = 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}y\right)$

$= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\cdot\frac{4\sqrt{2}}{3}\right) = 6 + \log_{3/2}\left(\frac{2}{3}\right)^2$

$= 6 - 2 = 4$