UNIT # 06

INDEFINITE & DEFINITE INTEGRATION

INDEFINITE INTEGRATION

EXERCISE - 01 — CHECK YOUR GRASP

1. $f(x) = \int \frac{2\sin x - \sin 2x}{x^3}$, $x \neq 0$

$\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} \frac{2\sin x}{x}\left(\frac{1-\cos x}{x^2}\right) = 1$

4. $\int \left(\frac{\cos^8 x - \sin^8 x}{1 - 2\sin^2 x\cos^2 x}\right) dx$

$= \int \frac{(\cos^4 x + \sin^4 x)(\cos^2 x - \sin^2 x)}{(1 - 2\sin^2 x \cos^2 x)} dx$

$= \int (\cos^2 x - \sin^2 x)\, dx = \int \cos 2x dx = \frac{\sin 2x}{2} + C$

5. $\int \frac{x^{1/3}}{(x^4-1)^{4/3}} dx = \frac{1}{4}\int \frac{4x^{-5}}{(1-x^{-4})^{4/3}} dx$

$= \frac{-3}{4}(1-x^{-4})^{-1/3} + C$ (Put $1 - x^{-4} = t$)

9. $\int \frac{(x^4+1)}{x(x^4+1+2x^2)} dx = \int \left(\frac{-2x}{x^4+1+2x^2} + \frac{1}{x}\right) dx$

$= \ell n|x| - \int \left(\frac{2x}{(x^2+1)^2}\right) dx = \ell n|x| + \frac{1}{x^2+1} + c$

10. $\int \left(\frac{x}{(1+x^5)}\right)^{3/2} dx = \int \frac{x^{-6}}{(1+x^{-5})^{3/2}} dx$

{Let $t = 1 + x^{-5}$; $dt = -5x^{-6}dx$}

$= -\frac{1}{5}\int \frac{dt}{(t)^{3/2}} = \frac{2}{5}(1+x^{-5})^{-1/2} + C$

13. $\int \left(\frac{x}{\sqrt{1+x^2}}\right) \ell n(x + \sqrt{1+x^2}) dx$

II I

$= \left[\sqrt{1+x^2}\,\ell n(x+\sqrt{1+x^2}) - \int \sqrt{1+x^2}\cdot\frac{1}{\sqrt{1+x^2}} dx\right]$

$= \sqrt{1+x^2}\,\ell n(x+\sqrt{1+x^2}) - x + C$

14. Let $\frac{1}{(x+2)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$

On solving it we get $A = -\frac{1}{5}, B = \frac{2}{5}, C = \frac{1}{5}$

$\therefore \int \left[\frac{-1}{5}\frac{x}{x^2+1} + \frac{2}{5}\frac{1}{(x^2+1)} + \frac{1}{5(x+2)}\right] dx$

$= \frac{-1}{10}\ell n(x^2+1) + \frac{2}{5}\tan^{-1} x + \frac{1}{5}\ell n|x+2| + K$

17. $\int \frac{x^4 - 4}{x^3\sqrt{\frac{4}{x^2} + 1 + x^2}} dx = \int \frac{x - 4x^{-3}}{\sqrt{4x^{-2} + 1 + x^2}} dx$

$= \frac{1}{2}\int t^{-1/2} dt$ [put $t = 4x^{-2} + 1 + x^2$]

$= \sqrt{\frac{4}{x^2} + 1 + x^2} + C = \frac{\sqrt{4 + x^2 + x^4}}{x} + C$

20. $\sqrt{1 + 2\tan x \sec x + 2\tan^2 x} = |\sec x + \tan x|$

$\therefore \int |\sec x + \tan x|\, dx = I$

$I = \ell n|\sec x + \tan x| + \ell n|\sec x| + c$ .....(i)

$I = \ell n|\sec^2 x + \sec x \tan x| + c$

$I = -\ell n|\sec x - \tan x| + \ell n|\sec x| + c$

$I = \ell n|1 + \tan x(\sec x + \tan x)| + c$

EXERCISE-02 — BRAIN TEASERS

2. $\int \frac{(\sin x + \sin 3x) + 3(\sin 3x + \sin 5x) + 3(\sin 5x + \sin 7x)}{\sin 2x + 3\sin 4x + 3\sin 6x} dx$

$= \int \frac{(2\sin 2x\cos x + 6\sin 4x \cos x + 6\sin 6x\cos x)}{\sin 2x + 3\sin 4x + 3\sin 6x} dx$

$= \int \frac{2\cos x(\sin 2x + 3\sin 4x + 3\sin 6x)}{\sin 2x + 3\sin 4x + 3\sin 6x} dx$

$= 2\sin x + c$

5. $I = \int \sin^2(\ell n x)\, dx$,

let $t = \ell n x \Rightarrow dt = \frac{dx}{x} \Rightarrow dx = e^t dt$

$\Rightarrow I = \int e^t \sin^2 t\, dt$

$I = \frac{1}{2}\int e^t(1 - \cos 2t)\, dt$

$\Rightarrow\ 2I = e^t - \int e^t \cos 2t\, dt$

Let $I_1 = \int e^t \cos 2t\, dt$

$= \frac{e^t}{5}\left[\cos 2t + 2\sin 2t\right] + C$

$\left(\because \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\right)$

$\Rightarrow\ I = \frac{1}{10} e^t(5 - 2\sin 2t - \cos 2t) + C$

$= \frac{x}{10}(5 - 2\sin(2\ell nx) - \cos(2\ell nx)) + C$

11. $\int \frac{3x^4-1}{(x^4+x+1)^2}dx = -\int\left(\frac{1}{x^4+x+1} - \frac{x(4x^3+1)}{(x^4+x+1)^2}\right)dx$

$= \frac{-x}{x^4+x+1} + c$

$\left(\int\left[f(x) + xf'(x)\right]dx = xf(x) + C\right)$

12. $\int \frac{dx}{[x^6(1-x^{-5})]^{1/3}} = \int \frac{x^{-6}}{(1-x^{-5})^{1/3}}\, dx$

$= \frac{1}{5}\int \frac{(1-x^{-5})^{2/3}}{2/3}dx = \frac{3}{10}\left(\frac{x^5-1}{x^5}\right)^{2/3} + c$

13. $I = \int \frac{\sin x}{\sin 4x}dx = \frac{1}{4}\int \frac{dx}{\cos x.\cos 2x}$

$= \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}dx$

Putting $t = \sin x$, we get $dt = \cos x.dx$

$I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}\, dt = \frac{1}{4}\int\left(\frac{2}{1-2t^2} - \frac{1}{1-t^2}\right)dt$

$= \frac{1}{4}\int \frac{dt}{\frac{1}{2}-t^2} - \frac{1}{4}\int \frac{dt}{1-t^2}$

$= \frac{1}{4}\frac{1}{\sqrt{2}}\ell n\left|\frac{\frac{1}{\sqrt{2}}+\sin x}{\frac{1}{\sqrt{2}}-\sin x}\right| - \frac{1}{8}\ell n\left|\frac{1+\sin x}{1-\sin x}\right| + C$

14. $\int \frac{d\theta}{\cos^3\theta\sqrt{\sin 2\theta}} = \int \frac{d\theta}{\cos^4\theta\sqrt{2\tan\theta}}$

$= \int \frac{\sec^2\theta(1+\tan^2\theta)}{\sqrt{2\tan\theta}}d\theta$

$= \frac{1}{\sqrt{2}}\int \frac{1+t^2}{\sqrt{t}}dt$ [put $\tan\theta = t$]

$= \frac{1}{\sqrt{2}}\left(2\sqrt{t} + \frac{t^{5/2}}{5/2}\right) = \frac{\sqrt{2}}{5}\sqrt{\tan\theta}\left(5 + \tan^2\theta\right) + c$

17. $I = \int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}\, dx = \int \frac{(1-\sqrt{x})}{\sqrt{x}(1-x)^{3/2}}\, dx$

$= \int \frac{dx}{x^2\left(\frac{1}{x}-1\right)^{3/2}} - \int \frac{dx}{(1-x)^{3/2}}$

$= \frac{2\sqrt{x}}{\sqrt{1-x}} - \frac{2}{\sqrt{1-x}} + C = \frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + C$

18. $f'(x) = 3x^2.\sin\frac{1}{x} - x.\cos\frac{1}{x}$

$\Rightarrow\ f(x) = \int\left(3x^2.\sin\frac{1}{x} - x\cos\frac{1}{x}\right)dx$

$= x^3\sin\frac{1}{x}. - \int \cos\frac{1}{x}\left(-\frac{1}{x^2}\right)x^3\, dx - \int x\cos\frac{1}{x}\, dx$

$= x^3\sin\frac{1}{x} + C$

since $f\left(\frac{1}{\pi}\right) = 0 + C \Rightarrow C = 0$

$\Rightarrow\ f(x) = \begin{cases} x^3\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f(x)$ is clearly continuous and differentiable at $x = 0$ zero with $f'(0) = 0$.

$f''(0) = \lim_{h\to 0}\frac{3h^2\sin\frac{1}{h} - h\cos\frac{1}{h}}{h}$

$= 3h\sin\frac{1}{h} - \cos\frac{1}{h}$

This limit does't exist, hence $f'(x)$ is non-differentiable at $x = 0$.

Also $\lim_{x\to 0} f'(x) = 0$. Thus $f'(x)$ is continuous at $x = 0$.

Fill in the blanks :

1. $\int \frac{4e^{2x}+6}{9e^{2x}-4}dx = Ax + B\log(9e^{2x}-4) + C$

Now differentiate both sides

$$\frac{4e^{2x}+6}{9e^{2x}-4} = A + \frac{B(18e^{2x})}{9e^{2x}-4}$$

$$\Rightarrow \frac{4e^{2x}+6}{9e^{2x}-4} = \frac{9Ae^{2x}+18Be^{2x}-4A}{9e^{2x}-4}$$

on comparing we get

$$A = \frac{-3}{2}, B = \frac{35}{36}, C \in R$$

2. An antiderivative of f(x)= F(x)

$= \int (\log(\log x) + (\log x)^{-2}\,dx + C$

$= x\log(\log x) - \int \frac{x}{x\log x}dx + \int (\log x)^{-2}\,dx + C$

(integrating by parts the first term)

$= x\log(\log x) - [x(\log x)^{-1} + \int (\log x)^{-2}\,dx]$

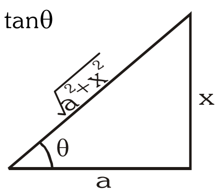
$+ \int (\log x)^{-2}\,dx + C$ (again integrating by parts)

$= x\log(\log x) - x(\log x)^{-1} + C$

Putting x = e, we have 1998 – e = e.0 + e + C. Thus C = 1998

Match the column :

2. (A) $\int \frac{dx}{(a^2+x^2)^{3/2}}$, put $x = a\tan\theta$



$$= \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\sin\theta$$

$$= \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

(B) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\int\left(\frac{-x^2+a^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}}\right)dx$

$$= -\int\sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

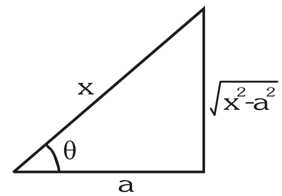
$$= -\frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + a^2\sin^{-1}\frac{x}{a}$$

$$= -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

(C) $\int \frac{dx}{(x^2-a^2)^{3/2}}$ Put $x = a\sec\theta$

$$= \int \frac{a\tan\theta\sec\theta}{a^3(\tan^3\theta)}d\theta = \frac{1}{a^2}\int\frac{\cos\theta}{\sin^2\theta}d\theta$$

$$= \frac{1}{a^2}\left(\frac{-1}{\sin\theta}\right) + C$$



$$= c - \frac{x}{a^2\sqrt{x^2-a^2}}$$

(D) $\int \frac{1}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) = \frac{1}{a}\cos^{-1}\left(\frac{a}{|x|}\right)$

$$= c + \frac{\pi}{2} - \frac{1}{a}\sin^{-1}\left(\frac{a}{|x|}\right)$$

$$= C - \frac{1}{a}\sin^{-1}\frac{a}{|x|}$$

Assertion & Reason :

1. Let $D(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x)$ where $\lambda_1 = (b_2c_3 - b_3c_2)$, $\lambda_2 = a_2c_3 - a_3c_2$, $\lambda_3 = a_2b_3 - a_3b_2$ then

$$\int D(x)\,dx = \int \lambda_1 f_1(x)\,dx + \int \lambda_2 f_2(x)\,dx + \int \lambda_3 f_3(x)\,dx + C \quad \ldots (1)$$

$$= \begin{vmatrix} \int f_1(x)dx & \int f_2(x)dx & \int f_3(x)dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + C$$

Thus statement-I is true and follows from statement-II which we have applies at Eq. (1)

3. The statement-II is false since in $\int \frac{dx}{x-3y}$

$= \log(x - 3y) + C$,

we are assuming that y is a constant. We will now prove the statement-I.

From the given relation $(x-y)^2 = \frac{x}{y}$, and

$2\log(x - y) = \log x - \log y \quad \ldots (1)$

Also, $\frac{dy}{dx} = \left(-\frac{y}{x}\right).\frac{x+y}{x-3y}$.

To prove the integral relation, it is sufficient to show that $\frac{d}{dx}$ RHS. $= \frac{1}{x-3y}$

Now, RHS $= \frac{1}{2}\log\left[\frac{x}{y}-1\right]$ $\left(\because (x-y)^2 = \frac{x}{y}\right)$

$= \frac{1}{2}[\log(x-y) - \log y]$

$= \frac{1}{2}\left[\frac{\log x - \log y}{2} - \log y\right]$ [From Eq. (1)]

$= \frac{1}{4}[\log x - 3\log y]$

$\Rightarrow \frac{d}{dx}$ RHS. $= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\frac{dy}{dx}\right]$

$= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\left(-\frac{y}{x}\right)\frac{x+y}{x-3y}\right] = \frac{1}{x-3y}$

Thus, statement-I is true.

Comprehension # 2 :

1. $I_n = \frac{x}{(x^2+a^2)^n} + 2n\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$

$= \frac{x}{(x^2+a^2)^n} + 2n\left[\int\left(\frac{x^2+a^2}{(x^2+a^2)^{n+1}} - \frac{a^2}{(x^2+a^2)^{n+1}}\right)dx\right]$

$= \frac{x}{(x^2+a^2)^n} + 2n\int\frac{1}{(x^2+a^2)^n}dx$

$-2na^2\int\frac{1}{(x^2+a^2)^{n+1}}dx$

$= \frac{x}{(x^2+a^2)^n} + 2n(I_n - a^2 I_{n+1})$

Whence $I_{n+1} + \frac{1-2n}{2n}\frac{1}{a^2} I_n = \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n}$

2. $I_{n,-m} = \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x} - \frac{n-1}{m-1}\int\frac{\sin^{n-2}x}{\cos^{m-2}x}dx$

$= \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x} - \frac{n-1}{m-1} I_{n-2,\,2-m}$

3. $u_{n+1} = \int\frac{x^{n+1}}{\sqrt{ax^2+2bx+c}}dx$

$= \frac{1}{2a}\int\frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+2bx+c}}dx$

$= \frac{1}{2a}\int\frac{x^n(2ax+2b)}{\sqrt{ax^2+2bx+c}}dx - \frac{b}{a}u_n$

$= I_n - \frac{b}{a}u_n$, where ... (i)

$I_n = \frac{1}{2a}\int\frac{x^n(2ax+2b)}{\sqrt{ax^2+2bx+c}}dx$

$= \frac{1}{2a}\, x^n 2\sqrt{ax^2+bx+c}$

$-\int nx^{n-1}2\sqrt{ax^2+2bx+c}\;dx$

$= \frac{x^n}{a}\sqrt{ax^2+2bx+c} -$

$\frac{n}{a}\int\frac{x^{n-1}(ax^2+2bx+c)}{\sqrt{ax^2+bx+c}}dx$ ...(ii)

from (i) and (ii) be get

$(n+1)a\,u_{n+1} + (2n+1)bu_n + ncu_{n-1}$

$= x^n\sqrt{ax^2+2bx+c}$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

5. $f'(x) = \frac{1}{1+x^2} + \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$

$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{1-x^4}$

Now $\int\frac{1}{2}\left(\frac{2}{1-x^4}\right)d(x^4) = \int\frac{4x^3}{1-x^4}dx$

$= -\ell n(1-x^4) + C$

8. Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$I = \int\frac{\tan^2\theta\sec^2\theta}{\left(\frac{\sin\theta}{\cos\theta}\sin(\tan\theta)+\cos(\tan\theta)\right)^2}d\theta$

$= \int\frac{\tan^2\theta}{\cos^2(\tan\theta-\theta)}d\theta$

Put $\tan\theta - \theta = u \Rightarrow \tan^2\theta d\theta = du$

$I = \int\frac{du}{\cos^2 u} = \int\sec^2 u\,du = \tan(\tan\theta-\theta)$

$= \tan(x - \tan^{-1}x) = \frac{\tan x - x}{1 + x\tan x}$

$= \frac{\sin x - x\cos x}{\cos x + x\sin x} + C$

9. $\int \cos 2\theta \ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

$= \ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2}.\frac{2}{\cos 2\theta}d\theta$

$= \frac{1}{2}\ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\sin 2\theta - \frac{1}{2}\ell n(\sec 2\theta)+C$

10. $I = \int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ell nx\ dx = \int\left(1+\frac{1}{\left(\frac{x}{e}\right)^{2x}}\right)\ell nx\left(\frac{x}{e}\right)^x dx$

$\left\{\text{Put}\left(\frac{x}{e}\right)^x = t \Rightarrow \ell nx\left(\frac{x}{e}\right)^x dx = dt\right\}$

$I = \int\left(1+\frac{1}{t^2}\right)dt = t-\frac{1}{t}+C = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C$

13. $\int \frac{3x^2+1}{(x^2-1)^3}dx = \int \frac{3x^2+1}{(x-1)^3(x+1)^3}dx$

$= \int -\frac{1}{2}\left[\frac{1}{(x+1)^3}-\frac{1}{(x-1)^3}\right]dx$

$= \frac{1}{2}\left[\frac{(x-1)^{-2}}{-2}+\frac{(x+1)^{-2}}{-2}\right] = C - \frac{x}{(x^2-1)^2}$

15. $\int \frac{\sec^2 x\,dx}{\tan^2 x+2\tan x} = \int \frac{dt}{t^2+2t}$ [Put tan x = t]

$= \frac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t+2}\right)dt = \frac{1}{2}\ell n\left|\frac{\tan x}{\tan x+2}\right|+C$

16. $\int (\sin x)^{-11/3}\cos x^{-1/3}dx = \int \frac{1}{(\sin^{11}x\cos x)^{1/3}}dx$

$= \int \frac{\operatorname{cosec}^4 x}{(\cot x)^{1/3}}dx = \int \frac{(1+\cot^2 x)\operatorname{cosec}^2 x}{(\cot x)^{1/3}}dx$

$= -\int (t^{-1/3}+t^{5/3})dt$ [Put cot x = t]

$= -\left[\frac{t^{2/3}}{2/3}+\frac{t^{8/3}}{8/3}\right]+C = \frac{-3}{8}[4\cot^{2/3}x+\cot^{8/3}x]+C$

19. $I = \int(\sqrt{\tan x}+\sqrt{\cot x})dx = \int \frac{\tan x+1}{\sqrt{\tan x}}dx$

put $\tan x = t^2$

$\Rightarrow (\sec^2 x)dx = 2tdt \quad \Rightarrow dx = \frac{2t}{1+t^4}dt$

$I = \int \frac{t^2+1}{\sqrt{t^2}}.\frac{2t}{t^4+1}dt = 2\int \frac{t^2+1}{t^4+1}dt$

$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt = 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{2}\right)^2}dt$

$= 2\int \frac{du}{u^2+\left(\sqrt{2}\right)^2}$, where $u = t - \frac{1}{t}$

$\Rightarrow \quad I = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$

$= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+C$

20. $I = \int \frac{\sqrt{\cos 2x}}{\sin x}dx$

$= \int \frac{\sqrt{\cos^2 x-\sin^2 x}}{\sin x}dx = \int \sqrt{\cot^2 x-1}\,dx$

Put $\cot x = \sec\theta$

$\Rightarrow \quad -\operatorname{cosec}^2 x\ dx = \sec\theta\ \tan\theta d\theta$

$\therefore \quad I = \int \sqrt{\sec^2\theta-1}.\frac{\sec\theta.\tan\theta}{-(1+\sec^2\theta)}d\theta$

$= -\int \frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}d\theta = -\int \frac{\sin^2\theta}{\cos\theta+\cos^3\theta}d\theta$

$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$

$= -\int \frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$

$= -\int \sec\theta d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}d\theta$

$= -\log|\sec\theta+\tan\theta| + 2\int \frac{\cos\theta}{2-\sin^2\theta}d\theta$

$= -\log|\sec\theta+\tan\theta| + 2\int \frac{dt}{2-t^2}$ (put $\sin\theta = t$)

$= -\log|\sec\theta+\tan\theta| + 2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|+C$

$= -\log|\cot x+\sqrt{\cot^2 x-1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+C$

4. $$\int \frac{\cot x dx}{(1-\sin x)(\sec x+1)} = \int \frac{\cos^2 x \, dx}{\sin x(1-\sin x)(1+\cos x)}$$

$$= \int \frac{(1+\sin x)}{\sin x(1+\cos x)}dx = \int \frac{dx}{\sin x(1+\cos x)} + \int \frac{1}{1+\cos x}dx$$

$$= \int \frac{\sin x \, dx}{(1-\cos^2 x)(1+\cos x)} + \tan\frac{x}{2} + C$$

$$= \int \frac{-dt}{(1+t)(1-t^2)} + \tan\frac{x}{2} + C$$

$$= \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + C$$

8. $$I = \int \frac{\ell n(\cos x + \sqrt{\cos 2x})}{\sin^2 x}dx$$

$$= \int \operatorname{cosec}^2 x \, \ell n\left[(\sin x)(\cot x + \sqrt{\cot^2 x - 1})\right]dx$$

$$= \int \ell n(\sin x)\operatorname{cosec}^2 x \, dx$$

$$+ \int \ell n(\cot x + \sqrt{\cot^2 x - 1}).\operatorname{cosec}^2 x \, dx$$

$$= -\ell n(\sin x)\cot x + \int \cot^2 x dx - \int \ell n(t + \sqrt{t^2-1})dt$$

[put cotx = t]

$$= -\ell n(\sin x)\cot x - \cot x - x - \left[\ell n\left(t + \sqrt{t^2-1}\right)\right]t$$

$$-\int t.\frac{1+\frac{2t}{2\sqrt{t^2-1}}}{t+\sqrt{t^2-1}}dt$$

$$= -\ell n(\sin x)\cot x - \cot x - x - t\ell n(t + \sqrt{t^2-1})$$

$$+\int \frac{t}{\sqrt{t^2-1}}dt$$

$$= -\ell n(\sin x)\cot x - \cot x - x - t\,\ell n(t + \sqrt{t^2-1}) + \sqrt{t^2-1}$$

$$= -\ell n(\sin x)\cot x - \cot x - x - \cot x\,\ell n\left(\cot x + \frac{\sqrt{\cos 2x}}{\sin x}\right) + \frac{\sqrt{\cos 2x}}{\sin x}$$

$$= \frac{\sqrt{\cos 2x}}{\sin x} - x - \cot x - \cot x\,\ell n(e(\cos x + \sqrt{\cos 2x})) + C$$

9. $$\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx = \int \frac{e^x(1+1-x^2)}{(1-x)\sqrt{1-x^2}}dx$$

$$= \int e^x\left(\underbrace{\frac{1}{(1-x)\sqrt{1-x^2}}}_{f'(x)} + \underbrace{\frac{\sqrt{1-x^2}}{1-x}}_{f(x)}\right)dx$$

$$= e^x\sqrt{\frac{1+x}{1-x}} + C$$

13. $$\int \frac{f(x)dx}{x^2(x+1)^3} = \int\left(\frac{1}{x^2} - \frac{x}{(x+1)^3}\right)dx$$

$$\int \frac{f(x)dx}{x^2(x+1)^3} = \int \frac{3x^2+3x+1}{x^2(x+1)^3}dx$$

$\Rightarrow f(x) = 3x^2 + 3x + 1$

$f'(0) = 3$

14. $$\int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x}dx$$

put cos x = t, we get

$-\sin x \, dx = dt$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\int e^t\left(\cos^{-1} t - \frac{1}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} + \frac{t}{(1-t^2)^{3/2}}\right)dt$$

$$= e^t\left(\cos^{-1} t + \frac{1}{\sqrt{1-t^2}}\right) + C$$

$= e^{\cos x}(x + \operatorname{cosec} x) + C$

15. $$\int \frac{x}{(7x-10-x^2)^{3/2}}dx = \int \frac{x}{[(x-5)(2-x)]^{3/2}}dx$$

(put $x = 5\cos^2\alpha + 2\sin^2\alpha$)

$$= \int \frac{(5\cos^2\alpha + 2\sin^2\alpha)(-3\sin 2\alpha)}{[(-3\sin^2\alpha)(-3\cos^2\alpha)]^{3/2}}d\alpha$$

$$= \int \frac{(5\cos^2\alpha + 2\sin^2\alpha)(-3\sin 2\alpha)}{27(\sin\alpha\cos\alpha)^3}d\alpha$$

$$= \frac{-6}{27}\int \frac{(5\cos^2\alpha + 2\sin^2\alpha)}{(\sin\alpha\cos\alpha)^2}d\alpha$$

$$= \frac{-2}{9}\int (5\operatorname{cosec}^2\alpha + 2\sec^2\alpha)d\alpha$$

$$= \frac{-2}{9}\left[-5\cot\alpha + 2\tan\alpha\right]$$

$$= \frac{-2}{9}\int (-5\cot\alpha + 2\tan\alpha)\ d\alpha$$

$$= \frac{10}{9}\sqrt{\frac{2-x}{x-5}} - \frac{4}{9}\sqrt{\frac{x-5}{2-x}} + C$$

EXERCISE - 05 [A] JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

8. $\int \frac{5\tan x}{\tan x - 2}dx = \int \frac{5\sin x}{\sin x - 2\cos x}dx$

5sinx = A(sinx – 2cosx) + B $\frac{d}{dx}$(sinx – 2cosx)

= A(sinx – 2cosx) + B (cosx + 2sinx)

$\left.\begin{matrix} A+2B=5 \\ -2A+B=0 \end{matrix}\right\}$ A = 1, B = 2

Now $\int \frac{5\sin x}{\sin x - 2\cos x}dx$

$= \int\left(\frac{(\sin x - 2\cos x)}{\sin x - 2\cos x} + \frac{2(\cos x + 2\sin x)}{\sin x - 2\cos x}\right)dx$

= x + 2log|sinx – 2cosx| + K

a = 2

9. put $x^3 = t \Rightarrow 3x^2dx = dt$

$I = \frac{1}{3}\int f(t)tdt$

$\frac{1}{3}\left[t\int f(t)dt - \int(1.\int f(t)dt)dt\right]$

$= \frac{1}{3}x^3\Psi(x^3) - \int \Psi(x^3)x^2dx$

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$= \int \sin^{-1}\left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}}\right]dx$

$= \int \sin^{-1}\left[\frac{x+1}{\sqrt{(x+1)^2+(3/2)^2}}\right]dx$

Put x + 1= 3/2 tanθ, $dx = \frac{3}{2}\sec^2\theta d\theta$

$= \int \sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta + \frac{9}{4}}}\right]\frac{3}{2}\sec^2\theta d\theta$

$= \frac{3}{2}\int \sin^{-1}\left[\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{1}\right]\sec^2\theta d\theta$

$= \frac{3}{2}\int \theta\sec^2\theta d\theta = \frac{3}{2}\left[\theta\tan\theta - \int \tan\theta d\theta\right]$

$\frac{3}{2}[\theta\tan\theta - \log|\sec\theta|] + C$

$I = \frac{3}{2}\left[\frac{2}{3}(x+1)\tan^{-1}\left[\frac{2}{3}(x+1)\right] - \log\sqrt{1+\frac{4}{9}(x+1)^2}\right] + C$

$=(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(9+4x^2+8x+4)$

$+ \frac{3}{4}\log 9 + C$

$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$

2. $I = \int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx$

$= \int (x^{3m}+x^{2m}+x^m)\left[\frac{2x^{3m}+3x^{2m}+6x^m}{x^m}\right]^{1/m}dx$

$= \int \left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$

$= \int (x^{3m-1}+x^{2m-1}+x^{m-1})(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$

Put $2x^{3m} + 3x^{2m} + 6x^m = y$

$\therefore I = \frac{1}{6m}\int y^{1/m}dy = \frac{1}{6m}\frac{y^{(1/m)+1}}{\left(\frac{m+1}{m}\right)} + C$

$= \frac{1}{6}\frac{y^{\frac{m+1}{m}}}{(m+1)} + C$

$= \frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1} + C$

4. Here $f\,f(x) = \frac{f(x)}{[1+f(x)^n]^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$

and $f\,f\,f(x) = \frac{x}{(1+3x^n)^{1/n}}$

$\therefore\ g(x) = (fofo......of)(x) = \dfrac{x}{(1+nx^n)^{1/n}}$

Let $I = \int x^{n-2} g(x)dx = \int \dfrac{x^{n-1}}{(1+nx^n)^{1/n}}dx$

$$= \frac{1}{n^2}\int \frac{n^2x^{n-1}}{(1+nx^n)^{1/n}}dx = \frac{1}{n^2}\int \frac{\frac{d}{dx}(1+nx^n)}{(1+nx^n)^{1/n}}dx$$

$$= \frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$$

6. $J - I = \int\left(\dfrac{e^{-x}}{e^{-4x}+e^{-2x}+1} - \dfrac{e^{x}}{e^{4x}+e^{2x}+1}\right)dx$

$$= \int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}dx = \int \frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1}dx$$

Let $e^x = t$ $\quad$ $e^x dx = dt$

$$= \int \frac{t^2-1}{t^4+t^2+1}dt = \int \frac{1-1/t^2}{(t+1/t)^2-1}dt$$

$$= \frac{1}{2}\ell n\left|\frac{t+\frac{1}{t}-1}{t+\frac{1}{t}+1}\right| + C = \frac{1}{2}\ell n\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right) + C$$

7. Let $I = \int \dfrac{\sec^2 x}{(\sec x + \tan x)^{9/2}}dx$

$$= \int \frac{\sec x(\sec x + \tan x)\sec x}{(\sec x + \tan x)^{11/2}}dx$$

Put $\sec x + \tan x = t$

$\Rightarrow (\sec x \tan x + \sec^2 x)\, dx = dt$

Also $\because \sec^2 x - \tan^2 x = 1$

$\Rightarrow \sec x - \tan x = \dfrac{1}{t}$

$\therefore \sec x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right)$

$$\therefore I = \frac{1}{2}\int \frac{\left(1+\frac{1}{t}\right)dt}{t^{11/2}} = \frac{1}{2}.\int\left(t^{-9/2} + t^{-13/2}\right)dt$$

$$= \frac{1}{2}\left(-\frac{2t^{-7/2}}{7} - \frac{2t^{-11/2}}{11}\right) + K$$

$$= -\frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) + K$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$