UNIT # 11

BINOMIAL THEOREM AND COMPLEX NUMBERS

BINOMIAL THEOREM

EXERCISE - 01 CHECK YOUR GRASP

3. $(1 - 2x + 5x^2)^n$

For sum of coefficients put x = 1

$\therefore \quad a = (1 - 2 + 5)^n = 4^n$

$(1 + x)^{2n}$

For sum of coefficients put x = 1;

$b = (1 + x)^{2n} = 2^{2n} = (2^2)^n = 4^n$

$\therefore \quad a = b$

5. Given expression can be rewritten as

$$\frac{2}{2^7\sqrt{4x+1}}\left[{}^7C_1(\sqrt{4x+1})+{}^7C_3(\sqrt{4x+1})^3+\ldots+{}^7C_7(\sqrt{4x+1})^7\right]$$

$$\frac{1}{2^6}\left[{}^7C_1+{}^7C_3(\sqrt{4x+1})^2+\ldots..+{}^7C_7(\sqrt{4x+1})^6\right]$$

$\therefore$ Last term becomes $(4x + 1)^3$

Hence degree is 3

7. $(4 + x + 7x^2)\left(x-\frac{3}{x}\right)^{11}$

$$\Rightarrow \quad 4\left(x-\frac{3}{x}\right)^{11}+x.\left(x-\frac{3}{x}\right)^{11}+7x^2\left(x-\frac{3}{x}\right)^{11}$$

term independent of x in above will be

4 coefficient of x^0 in expansion of $\left(x-\frac{3}{x}\right)^{11}$

+ 1 coefficient of x^{-1} in expansion of $\left(x-\frac{3}{x}\right)^{11}$

+ 7 coefficient of x^{-2} in expansion of $\left(x-\frac{3}{x}\right)^{11}$

$\because$ x^r in expansion of $\left(x-\frac{3}{x}\right)^{11}$

$$= {}^{11}C_r x^{11-r}\left(\frac{-3}{x}\right)^r \Rightarrow {}^{11}C_r \,.\, x^{11-2r} \,.\, (-3)^r$$

x^0 will not exist in expansion of $\left(x-\frac{3}{x}\right)^{11}$ for integral r.

x^{-1} will occur at r = 6

$\therefore$ coefficient of $x^{-1} = {}^{11}C_6(-3)^6 = 3^6 \,.\, {}^{11}C_6$

Also x^2 will not exist in expansion of $\left(x-\frac{3}{x}\right)^{11}$ for integral r.

$\therefore$ term independent of x in expansion will be

$= 3^6 .\, {}^{11}C_6$

8. Given $(1 + x)^n = a + b$ .... (1)

then, $(1 - x)^n = a - b$ .... (2)

Multiplying equation (1) & (2)

we get, $(1 - x^2)^n = a^2 - b^2$

10. Calculate $m=\frac{n+1}{1+\left|\frac{a}{b}\right|}$ as in $(a + b)^n$

$$m=\frac{13+1}{1+\left|\frac{2x}{5y}\right|}=\frac{14}{1+2}=\frac{14}{3}$$

m is not integer so greatest term is $T_{[m]+1}$

$T_5 = {}^{13}C_4(2x)^9.(5y)^4$

$= {}^{13}C_4 \,.\, 20^9 \,.\, 10^4$ $\quad[\because \; x = 10, y = 2]$

13. $\binom{47}{4}+\sum_{j=1}^{5}\binom{52-j}{3}=\binom{x}{y}$

L.H.S.

${}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$

Using property ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ we get

$= {}^{52}C_4 = \binom{x}{y} \Rightarrow x = 52, y = 4.$

15. Let $R'=(5\sqrt{5}-11)^{31}$

Now $\quad R - R' = (5\sqrt{5}+11)^{31}-(5\sqrt{5}-11)^{31}$

$\Rightarrow$ R – R' = Integer $\Rightarrow$ I + f – R' = Integer

$\Rightarrow$ f – R' is an Integer but $-1 < f - R' < 1$

so $f - R' = 0 \Rightarrow f = R'$

so $R.f = R.R' = (5\sqrt{5}+11)^{31}(5\sqrt{5}-11)^{31}$

$= 4^{31} = 2^{62}$

18. Let $b=\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-(n-r)}{{}^nC_r}$

$$=n\sum_{r=0}^{n}\frac{1}{{}^nC_r}-\sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$$

$$=na_n-\sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}} \qquad \because \; {}^nC_r={}^nC_{n-r}$$

$= na_n - b$

$\Rightarrow \quad 2b = na_n \Rightarrow b=\frac{na_n}{2}$

22. $(1 + x + x^2 + x^3)^{100}$

$= a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{300}x^{300}$ ....(i)

put x = 1 $\Rightarrow$ $4^{100} = a_0 + a_1 \ldots\ldots a_{300}$

So divisible by 2^{10}

Put $x = -1 \Rightarrow 0 = a_0 - a_1 + a_2 - .... + a_{300}$

$\Rightarrow \quad a_0 + a_2 + a_4 .... = a_1 + a_3 + ...$

replace x by $\frac{1}{x}$ we get $(x^3 + x^2 + x + 1)^{100}$

$= a_0x^{300} + a_1x^{399} + ...+a_{300}$ ....(ii)

by (i) & (ii)

$\Rightarrow a_0 = a_{300}$, $a_1 = a_{299}$,..... so 'B' is true

Diff. (i) we get $100(1 + x + x^2 + x^3)^{99} (1 + 2x + 3x^2)$

$= a_1 + 2a_2x + ... + 300x^{299}a_{300}$

put $x = 0 \Rightarrow 100 = a_1$

26. The number of term in the expansion of $(1 + x)^{2n}$ is $2n + 1$ (odd), its middle term is $(n + 1)^{th}$ term

$$\text{coefficient} = {}^{2n}C_n = \frac{2n!}{n!n!} = \frac{1.2.3.......(2n-1).2n}{n!n!}$$

$$= \frac{(1.3.5.......2n-1)(2.4.6.......2n)}{n!n!}$$

$$= \frac{(1.3.5.......2n-1)2^n(1.2.3.......n)}{n!n!}$$

$$= \frac{(1.3.5.......2n-1)2^n}{n!}$$

EXERCISE - 02 BRAIN TEASERS

1. Given expression is a G.P. therefore its sum is

$$(x+2)^{n-1}\left\{\frac{\left(\frac{x+1}{x+2}\right)^n - 1}{\left(\frac{x+1}{x+2}\right) - 1}\right\} = (x+2)^n - (x+1)^n$$

Now coefficient of $x^r = {}^nC_r2^{n-r} - {}^nC_r$

$= {}^nC_r(2^{n-r} - 1)$

3. $(1 - x + 2x^2)^{12} = {}^{12}C_0(1 - x)^{12} + {}^{12}C_1(1 - x)^{11}(2x^2)$

$+ {}^{12}C_2(1 - x)^{10}.4x^4 + {}^{12}C_3(1-x)^9\ 8x^6 + ...$

coeff. of $x^4 = {}^{12}C_4 + 2.{}^{12}C_1{}^{11}C_2 + 4.{}^{12}C_2$

$$= {}^{12}C_4 + 2.12.\frac{11.10}{2.1} + 4.{}^{12}C_2$$

$$= {}^{12}C_4 + 6\frac{12.11.10}{3.2.1} + 4.{}^{12}C_2$$

$= {}^{12}C_4 + 6{}^{12}C_3 + 4{}^{12}C_2$

$= {}^{12}C_4 + {}^{12}C_3 + {}^{12}C_3 + 4({}^{12}C_2 + {}^{12}C_3)$

$= {}^{12}C_3 + {}^{13}C_4 + 4.{}^{13}C_3 = {}^{12}C_3 + 3.{}^{13}C_3 + {}^{14}C_4$

4. $(1 + 2x^2 + x^4)(1 + x)^n = A_0 + A_1x + A_2x^2 + ...$

Here $A_0 = 1$, $A_1 = n$, $A_2 = 2 + {}^nC_2$

Given A_0, A_1, A_2 are in A.P.

$$\therefore n - 1 = 2 + \frac{n(n-1)}{2} - n$$

$\Rightarrow n^2 - 5n + 6 = 0 \Rightarrow n = 2, 3$

7. $(1 - x^{10})^{-1}(1 - x)^1$

$=(1+ x^{10}+ x^{20} +...+ x^{400}......)(1 - x)$

co-efficient of x^{401} is -1

11. Put $x = y = z$, we get $(1 + x)^{3n}$

$\therefore$ coeff. of terms of degree $r = {}^{3n}C_r$

14. $$\left(x^3 + 3.2^{-\log_2 x^3}\right)^{11} = \left(x^3 + \frac{3}{x^3}\right)^{11}$$

$$T_{r+1} = {}^{11}C_r(x^3)^{11-r}\left(\frac{3}{x^3}\right)^r = {}^{11}C_r(x)^{33-6r}(3)^r$$

Now $33 - 6r = 2 \Rightarrow 6r = 31$ (not possible)

$33 - 6r = -3 \Rightarrow r = 6$

$33 - 6r = 3 \Rightarrow r = 5$

$$\therefore \quad \frac{\text{coeff. of } x^3}{\text{coeff. of } x^{-3}} = \frac{{}^{11}C_5 3^5}{{}^{11}C_6 3^6} = \frac{1}{3}$$

15. $T_r = (r^2 + 1)r! = [r(r + 1) - (r - 1)]r!$

$= r((r + 1)!) - (r - 1)(r !)$

$\therefore T_n = n((n + 1)!) - (n - 1)(n!)$

$T_{n-1} = (n - 1)(n!) - (n - 2)((n - 1)!)$

$T_{n-2} = (n - 2)((n - 1)!) - (n - 3)((n - 2)!)$

....

....

$T_1 = 1.(2!) - 0$

adding all we get $S = n((n + 1)!)$

16. $$T_{r+1} = {}^{3n}C_r(x^K)^{3n-r}\left(\frac{1}{x^{2K}}\right)^r$$

$= {}^{3n}C_r\ x^{3nK - 3rK} = {}^{3n}C_r\ x^{3K(n-r)}$

Here $r \le n$

Hence $(n + 1)^{th}$ term is always independent of K.

18. ${}^{18}C_{r-2} + {}^{18}C_{r-1} + {}^{18}C_{r-1} + {}^{18}C_r \ge {}^{20}C_{13}$

$\Rightarrow {}^{19}C_{r-1} + {}^{19}C_r \ge {}^{20}C_{13}$

$\Rightarrow {}^{20}C_r \ge {}^{20}C_{13} \Rightarrow \quad {}^{20}C_r \ge {}^{20}C_7$

$\Rightarrow r = 7, 8, 9, 10, 11, 12, 13$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Fill in the Blanks :

1. 2^n will be the sum of all coefficients

$\therefore \quad 2^n = 4096$

$\Rightarrow \quad n = 12$

Hence greatest coefficient will be $^{12}C_6$.

2. $7^{1995} = 7[(7)^2]^{997} = 7(50 - 1)^{997}$

using expansion

$7\{^{997}C_0.50^{997} + {}^{997}C_1.50^{996} + \ldots + {}^{997}C_{996}.50 - 1\}$

$= 100K + 7(997 \quad 50 - 1)$

$= 100K + 7(900 \quad 50 + 7 \quad 50 - 1)$

$= 100K_1 + 49 \quad 50 - 7 = 100K_1 + 2450 - 7$

$= 100K_1 + 2400 + 43$

$\therefore$ Remainder is 43

Match the column :

1. (A) $(2n + 1)(2n + 3)(2n + 5) \ldots (4n - 1)$

$$= \frac{(2n!)(2n+1)(2n+2)(2n+3)(2n+4)\ldots(4n-1)(4n)}{(2n!)(2n+2)(2n+4)(2n+6)\ldots(4n)}$$

$$\frac{(4n!)(n!)}{(n!)(2n)!2^n(n+1)(n+2)\ldots(2n)}$$

$$= \frac{(4n!)(n!)}{2^n.(2n)!(2n)!}$$

(B) $$\sum_{r=1}^{n} r\frac{^nC_r}{^nC_{r-1}} = \frac{r.n!}{r!(n-r)!} \cdot \frac{(r-1)!(n-r+1)!}{n!}$$

$$= \sum_{r=1}^{n}(n - r + 1)$$

$$= n(n + 1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

(C) $(C_0+C_1)(C_1+C_2)(C_2+C_3)\ldots\ldots(C_{n-1}+C_n)$

$= mC_1C_2..C_{n-1}$

$$= \left(\frac{C_0}{C_1}+1\right)\left(\frac{C_1}{C_2}+1\right)\left(\frac{C_2}{C_3}+1\right)\ldots\left(\frac{C_{n-1}}{C_n}+1\right) = m$$

$$= \frac{n+1}{n} \cdot \frac{n+1}{n-1} \cdot \frac{n+1}{n-2} \ldots \frac{n+1}{1} = m$$

$$\frac{(n+1)^n}{n!} = m$$

(D) $$\sum_{i=1}^{n}\left\{\sum_{j=1}^{n} i\,C_iC_j + \sum_{j=1}^{n} j.C_iC_j\right\}$$

$$= \sum_{i=1}^{n} iC_i(2^n-1) + \sum_{i=1}^{n} n\,C_i.2^{n-1}$$

$$= n \,.\, 2^{n-1}(2^n - 1) + n \,.\, 2^{n-1}(2^n - 1)$$

$$= n \,.\, 2^n(2^n - 1)$$

Assertion & Reason :

1. Using expansion we get

$$\frac{(14!)}{r_1!\times r_2!\times r_3!\times r_4!}(a^{r_1}.b^{r_2}.c^{r_3}.d^{r_4})$$

where $r_1 + r_2 + r_3 + r_4 = 14$

$\Rightarrow \quad r_1 = 1,\ r_2 = 8,\ r_3 = 3,\ r_4 = 2$

$\therefore$ Coefficient of $ab^8c^3d^4$ is $\dfrac{14!}{1!\,8!\,3!\,2!}$

$\therefore$ Statement I is true & statement II explain I

Comprehension : # 1

1. $(1 + 4x + 4x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n} \,.\, x^{2n}$

$9^n = a_0 + a_1 + a_2 + \ldots.. + a_{2n}$ ......(i)

$1 = a_0 - a_1 + a_2 + \ldots.. + a_{2n}$ ......(ii)

adding (i) & (ii) we get

$$9^n + 1 = 2\sum_{r=0}^{n} a_{2r}$$

2. Subtracting (ii) from (i) we get

$$a^n - 1 = 2\sum_{r=1}^{n} a_{2r-1}$$

3. a_{2n-1} = coefficient of x^{2n-1} in

$(1 + 4x + 4x^2)^n = (1 + 2x)^{2n}$

$T_{r+1} = {}^{2n}C_r\,(2x)^r$

$a_{2n-1} = {}^{2n}C_{2n-1} \,.\, 2^{2n-1} = 2n \,.\, 2^{2n-1} = 2n \,.\, 2^{2n}$

4. $a_2 = {}^{2n}C_2 \,.\, 2^2 = 2n(2n - 1)^2 = 8n^2 - 4n$

4. LHS $= {}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{n-2}C_r + {}^{n-1}C_r$

$= {}^{r+1}C_{r+1} + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^{n-2}C_r + {}^{n-1}C_r$

$(\because\ {}^{r}C_r = {}^{r+1}C_{r+1})$

$S = {}^{r+2}C_{r+1} + {}^{r+3}C_{r+1} + \ldots = {}^{n}C_{r+1}$ $(\because\ {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r)$

5. $40x(1-x)^{39} + 2\times\frac{40}{2}x^2(1-x)^{35} + 3\times\frac{40}{3}x^3(1-x)^{37}$

$+\ldots 40.x^{40}$

$= 40x[(1-x)^{39} + x(1-x)^{38} + x^2(1-x)^{37} + \ldots x^{39}]$

$= 40x[(1-x)+x]^{39}] = 40x = ax + b$

$\Rightarrow$ a = 40 & b = 0

6. L.H.S.

${}^{n+1}C_2 + 2({}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + \ldots + {}^{n}C_2)$

$= {}^{n+1}C_2 + 2({}^{3}C_3 + {}^{3}C_2 + {}^{4}C_2 + \ldots + {}^{n}C_2)$

$= {}^{n+1}C_2 + 2.{}^{n+1}C_3$ $(\because\ {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r)$

$= {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_3$

$= {}^{n+2}C_3 + {}^{n+1}C_3$

$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} = \frac{n(n+1)(2n+1)}{6}$

L.H.S. = R.H.S.

$\frac{n(n+1)(2n+1)}{6} = \frac{100(101)(201)}{6}$

$\Rightarrow$ n = 100

7. $(101)^{50} - (99)^{50} = (1+100)^{50} + (1-100)^{50}$

$=2\{{}^{50}C_1 . 100 + {}^{50}C_3 . 100^3 + \ldots + {}^{50}C_{49} . 100^{49}\}$

=a positive quantity + $(100)^{50}$

Hence $(101)^{50} - (99)^{50} > 100^{50}$

$\therefore$ $(101)^{50} > (100)^{50} + (99)^{50}$

13. $S = {}^{n}C_0.\sin(0x).\cos nx + {}^{n}C_1.\sin x.\cos(n-1)x$

$\ldots\ldots + {}^{n}C_n \sin nx.\cos(0x)$ ...... (1)

$S = {}^{n}C_0.\sin nx.\cos(0x) + {}^{n}C_1.\sin(n-1)x.\cos x +$

$\ldots\ldots + {}^{n}C_n \sin(0x)\cos nx$ ...... (2)

Add (1) & (2)

$2S = ({}^{n}C_0 + {}^{n}C_1 + \ldots + {}^{n}C_n)\sin nx \Rightarrow S = 2^{n-1}\sin nx$

14. (c) $(a-b-c+d)^{10}$

$= \frac{10!}{r_1!.\ r_2!.\ r_3!.\ r_4!}\ a^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$

$= \frac{10!(-1)^3(-1)^4}{2!\times 3!\times 4!\times 1!} = -12600$

16. L.H.S.

${}^{n}C_r . {}^{r}C_k$

$\frac{n!}{r!(n-r)!}.\frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-k)!}.\frac{(n-k)!}{(r-k)!(n-r)!}$

$= {}^{n}C_k.{}^{n-k}C_{r-k} = \binom{n}{k}.\binom{n-k}{r-k}$

18. $\sum_{r=0}^{n-2}\binom{n-1}{r}\binom{n}{r+2} = \binom{2n-1}{n-2}$

L.H.S. $\sum_{r=0}^{n-2} {}^{n-1}C_r.{}^{n}C_{r+2}$

$\sum_{r=0}^{n-2} {}^{n-1}C_{n-r-1}.{}^{n}C_{r+2}$

Coefficient of x^{n+1} in the expansion of $(1+x)^{n-1}(1+x)^n$ i.e. $(1+x)^{2n-1}$

$= {}^{2n-1}C_{n+1} = {}^{2n-1}C_{n-2} = \binom{2n-1}{n-2}$

20. We know that

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_2x^r + \ldots + C_nx^n$ ....(i)

$\left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + \ldots$

$+ C_r\frac{1}{x^r} + \ldots + C_n\frac{1}{x^n}$ ....(ii)

Multiply (i) & (ii) we get

$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + \ldots C_rx^r + \ldots + C_nx^n)$

$\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_n}{x^n}\right)$ ....(iii)

Now coefficient of $\frac{1}{x^r}$ in RHS

$= C_0C_r + C_1C_{r+1} + \ldots C_{n-r}C_n$

$\therefore$ Coefficient of $\frac{1}{x^r}$ in LHS

= coefficient of x^{n-r} in $(1+x)^{2n} = {}^{2n}C_{n-r}$

$= \frac{2n!}{(n-r)!(n+r)!}$

But (iii) is an identity

$\therefore$ Coefficient of $\frac{1}{x^r}$ in RHS

= coefficient of $\frac{1}{x^r}$ in LHS

$\Rightarrow$ $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}$

25. $S = \sum_{r=0}^{n}\frac{{}^{n}C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1} = \frac{2^{n+1}-1}{n+1}$

26. $S = \sum_{r=1}^{n}\frac{r\ {}^{n}C_r}{{}^{n}C_{r-1}} = \sum_{r=1}^{n}\frac{r.n!}{r!(n-r)!}.\frac{(r-1)!(n-r+1)!}{n!}$

$= \sum_{r=1}^{n}(n - r + 1) = (n)(n + 1) - \frac{(n)(n+1)}{2}$

$= \frac{(n)(n+1)}{2}$

27. $\frac{1}{^{2n+1}C_r} + \frac{1}{^{2n+1}C_{r+1}} = \frac{r!(2n+1-r)!}{(2n+1)!} + \frac{(r+1)!(2n-r)!}{(2n+1)!}$

$= \frac{r!(2n-r)!}{(2n+1)!}\{2n+1-r+r+1\}$

$= \frac{2n+2}{2n+1}\cdot\frac{r!(2n-r)!}{2n!} = \frac{2n+2}{2n+1}\cdot\frac{1}{^{2n}C_r}$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. $\because \sum_{r=0}^{2n} a_r(x - 2)^r = \sum_{r=0}^{2n} b_r(x - 3)^r$

Let $y = x - 3 \Rightarrow y + 1 = x - 2$

so the given expression reduces to :

$\sum_{r=0}^{2n} a_r(1 + y)^r = \sum_{r=0}^{2n} b_r \cdot y^r$

$\Rightarrow a_0 + a_1(1 + y) + a_2(1 + y)^2 + \dots a_{2n}(1 + y)^{2n}$
$= b_0 + b_1 y + \dots\dots\dots + b_{2n} \cdot y^{2n}$

using $a_{k=1}$ for all $k \geq n$, then we get

$\Rightarrow a_0 + a_1(1 + y) + a_2(1 + y)^2 + \dots + a_{n-1}(1+y)^{n-1} + (1+ y)^n + (1 + y)^{n+1} + \dots + (1+y)^{2n}$
$= b_0 + b_1 y + \dots + b_n y^n + \dots + b_{2n} \cdot y^{2n}$

Compare the co-efficients of y^n on both sides we get

$^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots\dots + {}^{2n}C_n = b_n$

$\Rightarrow {}^{n+1}C_{n+1} + {}^{n+1}C_n + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$

(use $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

$^{n+2}C_{n+1} + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$

(adding the first two terms).

$\Rightarrow$ If we combine terms on LHS, finally we get

$^{2n+1}C_{n+1} = b_n$

4. $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$ ....(i)

Put $x \to \frac{1}{x}$

$\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 + a_1\cdot\frac{1}{x} + a_2\cdot\frac{1}{x^2} + \dots + a_{2n}\cdot\frac{1}{x^{2n}}$

$(x^2 + x + 1)^n = a_0 . x^{2n} + a_1 . x^{2n-1} + \dots + a_{2n}$ ...(ii)

from (i) & (ii) we get

$\Rightarrow a_0 = a_{2n},\ a_1 = a_{2n-1}$ & so on.

Put $x \to -x$

$(1 - x + x^2)^n = a_0 - a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$

(a) $S = a_0a_1 - a_1a_2 + a_2a_3 + \dots\dots$
$= a_0a_{2n-1} - a_1a_{2n-2} + \dots\dots$

coeff. of x^{2n-1} in $(1 + x + x^2)(1 - x + x^2)^n$
$= (1 + x^2 + x^4)^n = 0$

(b) $S_1 = a_0 . a_{2n-2} - a_1a_{2n-3} + \dots\dots$
$=$ Coefficient of x^{2n-2} in $(1 + x^2 + x^4)^n$

Put $x = x^2$ in (i)

$(1 + x^2 + x^4)^n = a_0 + a_1x^2 + \dots + a_{n-1}x^{2n-2} + a_{n+1} . x^{2n+2} + \dots$

$S_1 = a_{n+1} = a_{n-1}$

(c) $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$

Put $x = 1$; $3^n = a_0 + a_1 + a_2 + a_3 + a_4 + \dots$

Put $x = w$; $0 = a_0 + a_1w + a_2w^2 + a_3w + a_4w + \dots$

Put $x = w^2$; $0 = a_0 + a_1w^2 + a_2w + a_3 + a_4w^2 + \dots$

$3^n = 3(a_0 + a_3 + \dots)$

$\Rightarrow E_1 = 3^{n-1}$

Similarly E_2 & E_3

6. $S = \sum_{r=0}^{n}\frac{^nC_r . 2^{r+2}}{(r+1)(r+2)} = \frac{1}{n+1}\sum_{r=0}^{n}\frac{^{n+1}C_{r+1} . 2^{r+2}}{r+2}$

$= \frac{1}{(n+1)(n+2)}\sum_{r=0}^{n} {}^{n+2}C_{r+2} . 2^{r+2}$

$(\because (1 + 2)^{n+2} = {}^{n+2}C_0 + {}^{n+2}C_1 2^1 + \dots\dots \sum_{r=0}^{n} {}^{n+2}C_{r+2})$

$S = \frac{1}{(n+1)(n+2)}\{3^{n+2} - 1 - 2n - 4\} = \frac{3^{2n+2} - 2n - 5}{(n+1)(n+2)}$

12. $S = (^{2n}C_1)^2 + 2 . (^{2n}C_2)^2 + 3(^{2n}C_3)^2 + \dots + 2n(^{2n}C_{2n})^2$

$= \sum_{r=1}^{2n} r . (^{2n}C_r)^2 = \sum_{r=1}^{2n} r . {}^{2n}C_r . {}^{2n}C_r$

$\therefore S = \sum_{r=1}^{2n} 2n . {}^{2n-1}C_{r-1} . {}^{2n}C_{2n-r}$

S is coefficient of x^{2n-1} in expansion of $(1+x)^{4n-1}$

$S = 2n . {}^{4n-1}C_{2n-1} = 2n . \frac{(4n-1)!}{(2n-1)!(2n!)}$

$\therefore S = \frac{(4n-1)!}{[(2n-1)!]^2}$

1. By hypothesis, $2^n = 4096 = 2^{12} \Rightarrow n = 12$

Since n is even, hence greatest coefficient

$= {}^nC_{n/2} = {}^{12}C_6 = \dfrac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4\cdot5\cdot6} = 924$

6. Coefficient of x^n in expansion of $(1 + x)(1 - x)^n$ i.e., coefficient of x^n in expansion of $(1 - x)^n$ + coefficient of x^{n-1} in expansion of $(1 - x)^n$]

Now, $(-1)^n\,{}^nC_n + (-1)^{n-1}\,{}^nC_{n-1}$

$(-1)^n[{}^nC_n - {}^nC_{n-1}] = (-1)^n[1-n]$.

7. Coefficient of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in expansion of $(1 + x)^n$ are ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$

Then $2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$\Rightarrow n^2 - n(4r + 1) + 4r^2 - 2 = 0$

Short Method: Let r = 1, hence ${}^nC_0$, ${}^nC_1$ and ${}^nC_2$ are in A.P.

$\Rightarrow 2.{}^nC_1 = {}^nC_0 + {}^nC_2 \Rightarrow 2n = 1 + \dfrac{n(n-1)}{2}$

$\Rightarrow 4n = 2 + n^2 - n \Rightarrow n^2 - 5n + 2 = 0$

Which is given by (b)

8. In the expansion of $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$, the general term is

$T_{r+1} = {}^{11}C_r (ax^2)^{11-r}\left(\dfrac{1}{bx}\right)^r = {}^{11}C_r a^{11-r}\dfrac{1}{b^r}x^{22-3r}$

For x^7, we must have $22 - 3r = 7 \Rightarrow r = 5$, and the

coefficient of $x^7 = {}^{11}C_5.a^{11-5}\dfrac{1}{b^5} = {}^{11}C_5\dfrac{a^6}{b^5}$.

Similary, in the expansion of $\left(ax^2 - \dfrac{1}{bx}\right)^{11}$, the

general term is $T_{r+1} = {}^{11}C_r(-1)^r\dfrac{a^{11-r}}{b^r}x^{11-3r}$

For x^{-7} we must have $11 - 3r = -7 \Rightarrow r = 6$, and

the coefficient of x^{-7} is ${}^{11}C_6\dfrac{a^5}{b^6} = {}^{11}C_5\dfrac{a^5}{b^6}$

As given, ${}^{11}C_5\dfrac{a^6}{b^5} = {}^{11}C_5\dfrac{a^5}{b^6} \Rightarrow ab = 1$.

10. $s = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 \ldots\ldots + {}^{20}C_{10}$

$s = {}^{20}C_{10} - {}^{20}C_{11} + \ldots\ldots\ldots + {}^{20}C_{20}$

$2s = 0 + {}^{20}C_{10}$

$s = \dfrac{1}{2}({}^{20}C_{10})$

13. It is obvious that remainder left out is 2.

14. $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j = \sum_{j=1}^{10} j^2 \times \dfrac{10}{j}\times{}^9C_{j-1}$

$S_3 = 10\left\{\sum_{j=1}^{10}(j-1+1)\dfrac{9}{j-1}\,{}^8C_{j-2}\right\}$

$= 10\left\{\sum_{j=2}^{10}9.{}^8C_{j-2} + \sum_{j=1}^{10}{}^9C_{j-1}\right\}$

$S_3 = 10\,(9\,.\,2^8 + 2^9)$

$10\,\{2^8\,(11)\} = 110 \quad 2^8 = 55 \quad 2^9$

so statement-1 is true

statement-2

$S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j = 10\sum_{j=1}^{10}{}^9C_{j-1} = 10.2^9$

so, statement-2 is wrong.

15. $(1 - x - x^2 + x^3)^6 \Rightarrow [(1 - x) - x^2(1 - x)]$

$\Rightarrow (1 - x)^6 (1 - x^2)^6$

$\Rightarrow (1 - {}^6C_1 x + {}^6C_2x^2 - {}^6C_3x^3 + {}^6C_4x^4 - {}^6C_5 + {}^6C_6x^6) \quad (1 - {}^6C_1x^2 + {}^6C_2x^4 - {}^6C_3x^6 + \ldots.)$

$\Rightarrow$ cofficient of x^7 is $\rightarrow$

${}^6C_1.\,{}^6C_3 - {}^6C_3.\,{}^6C_2 + {}^6C_5\,.\,{}^6C_1$

$\Rightarrow \dfrac{6.5.4}{3.2} - \dfrac{6.5.4}{3.2.1} + 6.6 = 120 - 300 + 36$

$\Rightarrow -144$

16. $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$

$= 2[T_2 + T_2 + T_6 + \ldots\ldots\ldots + T_{2n}]$

$= 2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + \ldots\ldots\ldots]$

= An Irrational Number

17. $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1} - \dfrac{x-1}{x-\sqrt{x}}\right)^{10}$

$\left(x^{1/3}+1-\left(\dfrac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$

$\left(x^{1/3}-x^{-1/2}\right)^{10}$

$T_{r+1} = {}^{10}C_r\left(x^{1/3}\right)^{10-r}\left(x^{-1/2}\right)^r$

$\dfrac{10-r}{3} - \dfrac{r}{2} = 0$

$20 - 2r = 3r$

$r = 4$

$T_5 = T_{4+1} = {}^{10}C_4 = \dfrac{10!}{6!.4!} = 210$

1. (a) $S = {}^nC_r + {}^nC_{r-1} + {}^nC_{r-1} + {}^nC_{r-2}$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$

($\therefore$ using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

(b) $(a-b)^n \Rightarrow$ general term $T_{r+1} = {}^nC_r \cdot a^{n-r} \cdot b^r(-1)^r$

${}^nC_4 \cdot a^{n-4} \cdot b^4 - {}^nC_5 \cdot a^{n-5} \cdot b^5 = 0$

$${}^nC_4\left(\frac{a}{b}\right) = {}^nC_5 \Rightarrow \frac{n!}{4!(n-4)!}\left(\frac{a}{b}\right) = \frac{n!}{5!(n-5)!}$$

$$\left(\frac{a}{b}\right) = \frac{(n-4)}{5}$$

2. Given that for positive integer m and n such that $n \ge m$.

LHS $= {}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots\ldots + {}^nC_m$

(in reverse order)

$= ({}^{m+1}C_{m+1} + {}^{m+1}C_m) + {}^{m+2}C_m + \ldots . {}^{n-1}C_m + {}^nC_m$

$[\because {}^mC_m = {}^{m+1}C_{m+1}]$

$({}^{m+2}C_{m+1} + {}^{m+2}C_m) + {}^{m+3}C_m + \ldots . + {}^nC_m$

$[\because {}^nC_{r+1} + {}^nC_r = {}^{n+1}C_{r+1}]$

$= {}^{m+3}C_{m+1} + {}^{m+3}C_m + \ldots . + {}^nC_m$

Combining in the same way we get

$= {}^nC_{m+1} + {}^nC_m = {}^{n+1}C_{m+1}$ = RHS

Again we have prove that

${}^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + \ldots . + (n-m+1)\,{}^mC_m = {}^{n+2}C_{m+2}$

LHS $= [{}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots . + {}^mC_m] +$

$[{}^{n-1}C_m + {}^{n-2}C_m + \ldots . + {}^mC_m] +$

$[{}^{n-2}C_m + \ldots\ldots + {}^mC_m] + \ldots . + [{}^mC_m]$

$(n-m+1)$ bracketed terms

$= {}^{n+1}C_{m+1} + {}^nC_{m+1} + {}^{n-1}C_{m+1} + \ldots . . + {}^{m+1}C_{m+1}$

(using previous result)

$= {}^{n+2}C_{m+2}$ = RHS

(Replace n by n + 1 & m by m + 1 in previous result).

3. $S = \sum_{i=0}^{m} {}^{10}C_i \cdot {}^{20}C_{m-i}$

S = coefficient of x^m in expansion of $(1 + x)^{30}$

$S = {}^{30}C_m$, maximized at m = 15.

4. (a) $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$

$= (1 + t^2)^{12}(1 + t^{12} + t^{24})$

$= (1 + t^2)^{12} + t^{12}(1 + t^2)^{12} + (1 + t^2)^{12} t^{24}$

$= 1 + {}^{12}C_6 + 1$

(b) $S = 2^k \cdot {}^nC_0 \cdot {}^nC_k - 2^{k-1}\,{}^nC_1 \cdot {}^{n-1}C_{k-1} + 2^{k-2}\,{}^nC_2 \cdot {}^{n-2}C_{k-2}$

$$S = \sum_{r=0}^{k} 2^{k-r}\,{}^nC_r\,{}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k} 2^{k-r}\frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{(k-r)!\,(n-k)!}$$

$$= \frac{n!}{(n-k)!\,k!}\sum_{r=0}^{k} 2^{k-r} \cdot \frac{k!}{r!(n-k)!}$$

$= {}^nC_k\,(2-1)^k = {}^nC_k$

5. ${}^{n-1}C_r = (k^2 - 3) \cdot {}^nC_{r+1}$

$$\frac{(n-1)!}{r!(n-r-1)!} = (k^2 - 3) \cdot \frac{n!}{(r+1)!(n-r-1)!}$$

$$\Rightarrow \frac{r+1}{n} + 3 = k^2 \Rightarrow k^2 - 3 > 0$$

$\Rightarrow k \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ ...(i)

$n - 1 \ge r$

$$\frac{r+1}{n} \le 1 \Rightarrow k^2 \le 3 + 1$$

$-2 \le k \le 2$ ... (ii)

$k \in [-2, -\sqrt{3}) \cup (\sqrt{3}, 2]$

7. $A_r = {}^{10}C_r$, $B_r = {}^{20}C_r$, $C_r = {}^{30}C_r$

$$\sum_{r=1}^{10}\left({}^{20}C_{10}\,{}^{10}C_r\,{}^{20}C_r - {}^{30}C_{10}\left({}^{10}C_r\right)^2\right)$$

$$= {}^{20}C_{10}\left({}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots . + {}^{10}C_{10}\,{}^{20}C_{10}\right)$$

$$-{}^{30}C_{10}\left({}^{10}C_1^2 + {}^{10}C_2^2 + \ldots + {}^{10}C_{10}^2\right)$$

$$= {}^{20}C_{10}\left({}^{30}C_{10} - 1\right) - {}^{30}C_{10}\left({}^{20}C_{10} - 1\right)$$

$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

8. Let the three consecutive terms be ${}^{n+5}C_{r-1}$, ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$

$$\therefore \frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r} = \frac{1}{2} \Rightarrow \frac{r}{n-r+6} = \frac{1}{2}$$

$\Rightarrow n = 3r - 6$ ....(1)

$$\text{Also, } \frac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \frac{5}{7} \Rightarrow \frac{r+1}{n-r+5} = \frac{5}{7}$$

$\Rightarrow 12r = 5n + 18$ ....(2)

Solving (1) and (2), we get n = 6