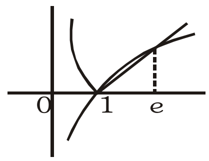
UNIT # 08

AREA UNDER THE CURVE AND DIFFERENTIAL EQUATIONS

AREA UNDER THE CURVE

EXERCISE - 01 — CHECK YOUR GRASP

3. $A = \int_1^e (\ell n\, x - \ell n^2 x)dx$



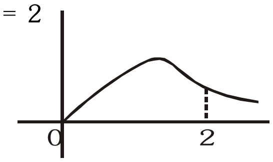
on solving it by parts we get

$A = 3x(\ell n\, x - 1)\Big|_1^e - x(\ell n^2 x)\Big|_1^e = 3 - e$

8. $y' = e^{-x} - xe^{-x} = e^{-x}(1 - x)$

$y'' = -e^{-x} - e^{-x}(1 - x) \Rightarrow -e^{-x}(2 - x) = 0 \Rightarrow x = 2$

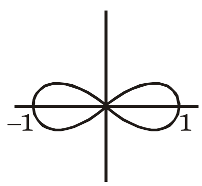
so point of inflection is x = 2



$A = \int_0^2 xe^{-x}\ dx$

$= 1 - 3e^{-2}$

10. $y = \pm\, x\sqrt{1 - x^2}$



$A = 4\int_0^1 x\sqrt{1 - x^2}\ dx$

$= -2\dfrac{(1 - x^2)^{3/2}}{3/2}\Bigg|_0^1 = \dfrac{4}{3}$

14. Given curve is

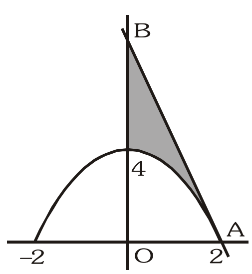
$y = f(x) = Ax^2 + Bx + C$ ....(i)

It passes through (1, 3)

$\therefore\ 3 = A + B + C$ ....(ii)

point (2, 0) also lie on curve

$\therefore\ 0 = 4A + 2B + C$ ....(iii)



from (i) $\dfrac{dy}{dx}\Big|_{(2,0)} = 4A + B$

slope of tangent is –4

$\therefore\ -4 = 4A + B$ ....(iv)

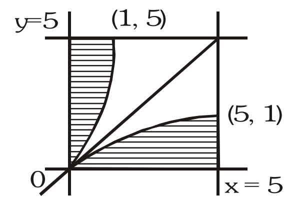
$\therefore$ (ii) (iii) & (iv) we get

A = –1, B = 0, C = 4

required curve is $y = -x^2 + 4$

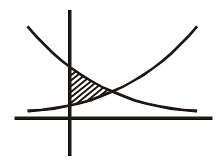
required area = area of $\Delta OAB - \int_0^2 (-x^2 + 4)dx$

15. $A = \int_0^1 (5 - 3x^3 - 2x)\ dx$



$= \dfrac{13}{4}$

17. $S = \int_0^{a/2} (e^{a-x} - e^x)dx$



$= -\ [2e^{a/2} - (e^a + 1)]$

Now $\lim\limits_{a \to 0} \dfrac{e^a - 2e^{a/2} + 1}{a^2} = \lim\limits_{a \to 0} \left(\dfrac{e^{a/2} - 1}{a/2}\right)^2 \dfrac{1}{4} = \dfrac{1}{4}$

20. The two curves meet at

$mx = x - x^2$ or $x^2 = x(1 - m)\ \therefore\ x = 0,\ 1 - m$

$\int_0^{1-m} (y_1 - y_2)dx = \int_0^{1-m} (x - x^2 - mx)dx = \left[(1 - m)\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^{1-m}$

$= \dfrac{9}{2}$ if m < 1

or $(1 - m)^3\left[\dfrac{1}{2} - \dfrac{1}{3}\right] = \dfrac{9}{2}$ or $(1-m)^3 = 27$

$\therefore\ m = -2$

But if m > 1 then 1 – m is negative, then

$\left[(1 - m)\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{1-m}^0 = \dfrac{9}{2}$

$-(1 - m)^3\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{9}{2}$

$\therefore\ -(1 - m)^3 = -27$ or $1 - m = -3\ \therefore\ m = 4.$

3. $\ell n f'(x) = x + c \quad \Rightarrow f'(x) = ke^x$

$\Rightarrow f'(x) = e^x \qquad \{f'(0) = 1 \Rightarrow k = 1\}$

$\Rightarrow f(x) = e^x + \lambda$

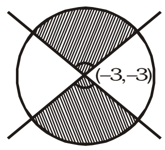
$\Rightarrow f(x) = e^x - 1 \qquad \{f(0) = 0 \Rightarrow \lambda = -1\}$

$$A = \int_0^1 (e^x - 1 + 1)\, dx = e - 1$$

6. $x^2 + y^2 + 6(x + y) + 2 \le 0$

& $x^2 - y^2 + 6(x - y) \le 0$

$\Rightarrow (x - y)(x + y + 6) \le 0$



from this we get a circle is two straight line which are at right angle

area = 2 quarter half circle = $\dfrac{\pi r^2}{2}$

7. $A = \displaystyle\int_a^{2a}\left(\frac{x}{6} + \frac{1}{x^2}\right)dx = \left(\frac{x^2}{12} - \frac{1}{x}\right)\Bigg|_a^{2a} = \frac{a^2}{4} + \frac{1}{2a}$

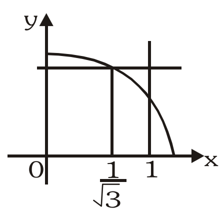
Now $f(a) = \dfrac{a^2}{4} + \dfrac{1}{2a}$

$\Rightarrow f'(a) = \dfrac{a}{2} - \dfrac{1}{2a^2}$

$f'(a) = 0 \Rightarrow a = 1$

$f''(a) > 0$ so at $a = 1$, $f(a)$ is minimum

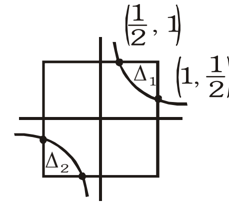
8. $A = \dfrac{1}{\sqrt{3}} + \displaystyle\int_{1/\sqrt{3}}^{1}\sqrt{\frac{4}{3} - x^2}\,dx$



$$= \frac{1}{\sqrt{3}} + \left[\frac{x}{2}\sqrt{\frac{4}{3} - x^2} + \frac{2}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{2}\right)\right]_{1/\sqrt{3}}^{1}$$

$$= \frac{1}{\sqrt{3}} + \left[\left(\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}}\right) + \frac{2}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right] = \frac{3\sqrt{3} + \pi}{9}$$

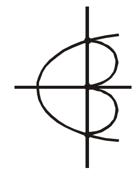
9. $\Delta_2 = \Delta_1 = \displaystyle\int_{1/2}^{1}\left[1 - \frac{1}{2x}\right]dx$



$= \dfrac{1}{2} - \dfrac{1}{2}\ \ell n\ 2$

$A = 4 - (\Delta_1 + \Delta_2) = 4 - (1 - \ell n\ 2) = 3 + \ell n\ 2$

11. $A = 2\displaystyle\int_0^1 [y\sqrt{1 - y^2} - (y^2 - 1)]\ dy$



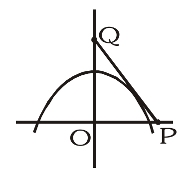
$$= \frac{-2}{3}(1 - y^2)^{3/2}\Big|_0^1 - \left(\frac{2y^3}{3} - 2y\right)_0^1 = 2$$

12. Equation of tangent is $y - 1 + \alpha^2 = -2\alpha(x - \alpha)$

so $P\left(\dfrac{1 + \alpha^2}{2\alpha}, 0\right)$ & $Q(0, \alpha^2 + 1)$

area of $\Delta OPQ(\Delta) = \dfrac{1}{2}\dfrac{(\alpha^2 + 1)^2}{2\alpha}$

$= \dfrac{1}{4}\left[\alpha^3 + 2\alpha + \dfrac{1}{\alpha}\right]$



$\Delta' = \dfrac{1}{4}\left[3\alpha^2 + 2 - \dfrac{1}{\alpha^2}\right] = 0$

$\Rightarrow 3\alpha^4 + 2\alpha^2 - 1 = 0 \quad \Rightarrow \alpha = \dfrac{1}{\sqrt{3}}$

so $\Delta = \dfrac{4}{3\sqrt{3}}$

Now $\dfrac{4}{3\sqrt{3}} = k\displaystyle\int_0^1 (1 - x^2)\ dx$

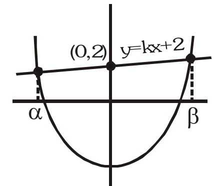
$\Rightarrow \dfrac{4}{3\sqrt{3}} = \dfrac{2k}{3}$

$\Rightarrow k = \dfrac{2}{\sqrt{3}}$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Assertion & Reason :

2. $A = \displaystyle\int_\alpha^\beta (kx + 2 - x^2 + 3)dx$



$$= \left(\frac{kx^2}{2} - \frac{x^3}{3} + 5x\right)_\alpha^\beta$$

$$= \left(\frac{k(\alpha + \beta)}{2} - ((\alpha + \beta)^2 - \alpha\beta)\frac{1}{3} + 5\right)(\beta - \alpha)$$

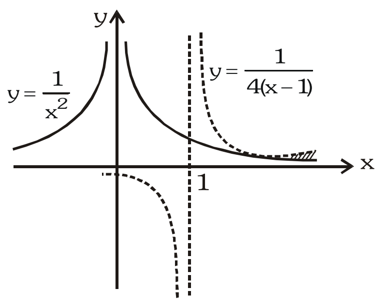
$$= \sqrt{k^2 + 20}\left[\frac{k^2}{2} - \left(\frac{k^2 + 5}{3}\right) + 5\right] = \frac{1}{6}(k^2 + 20)^{3/2}$$

Hence statement I is true & II is false.

Comprehension : # 1

Now $\displaystyle\int_2^a\left[\frac{1}{4(x - 1)} - \frac{1}{x^2}\right]dx = \frac{1}{a}$

$\Rightarrow a = e^2 + 1$



Also $\int_b^2\left[\frac{1}{4(x-1)}-\frac{1}{x^2}\right]dx = 1-\frac{1}{b}$

$\Rightarrow \left[\frac{1}{4}\ell n(x-1)+\frac{1}{x}\right]_b^2 = 1-\frac{1}{b}$

$\Rightarrow -\ell n\,(b-1) = 2 \quad \Rightarrow \quad b = 1+e^{-2}$

1. $\ell n\left(\frac{a}{b}\right) = \ell n\left(\frac{e^2+1}{1+e^{-2}}\right) = 2$

2. $|A| = \ell n\,(a-1)\,\ell n\,(b-1) = -4$

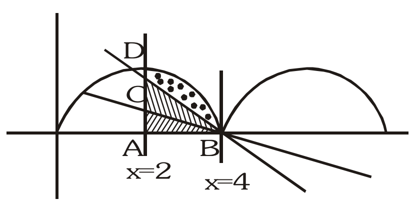
$A^{-1} = \frac{-1}{4}\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix} = \frac{A}{4}$

3. $z = 2-2i$

$\arg(z) = \frac{-3\pi}{4}$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

4. Let equation of line is $y = mx-4m$



$A = \int_2^4 \sqrt{2}\sin\frac{\pi}{4}x\,dx = \left[-\sqrt{2}\,\frac{4}{\pi}\cos\frac{\pi x}{4}\right]_2^4 = \frac{4\sqrt{2}}{\pi}$ ... (i)

Also area of $\Delta ABC = \frac{1}{2}.2.(-2m_1) = -2m_1$ ... (ii)

from (i) and (ii)

$-2m_1 = \frac{4\sqrt{2}}{3\pi} \Rightarrow m_1 = \frac{-2\sqrt{2}}{3\pi}$

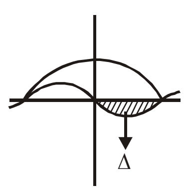
$\Rightarrow \tan(\pi-\theta_1) = \frac{-2\sqrt{2}}{3\pi} \quad \Rightarrow \quad \pi-\theta_1 = \tan^{-1}\frac{2\sqrt{2}}{3\pi}$

$\Rightarrow \theta_1 = \pi-\tan^{-1}\frac{2\sqrt{2}}{3\pi}$ or $\frac{1}{2}.(2)\,(-2m_2) = \frac{8\sqrt{2}}{3\pi}$

$\Rightarrow m_2 = \frac{-4\sqrt{2}}{3\pi} \quad \Rightarrow \quad \tan(\pi-\theta_2) = \frac{-4\sqrt{2}}{3\pi}$

$\Rightarrow \theta_2 = \pi-\tan^{-1}\frac{4\sqrt{2}}{3\pi}$

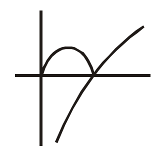
5. $\Delta = \left|\int_0^1 (x^3-x)dx\right| = \frac{1}{4}$



Area $= \frac{\pi}{4}+\frac{1}{4}+\frac{\pi}{4}-\frac{1}{4} = \frac{\pi}{2}$

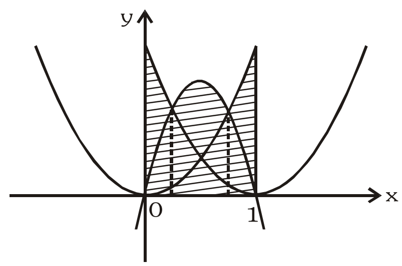
required ratio $= \frac{\frac{\pi}{4}+\frac{1}{4}}{\frac{\pi}{4}-\frac{1}{4}}$

8. $A = \int_0^1 (\sin^4\pi x-\ell n\,x)\,dx$



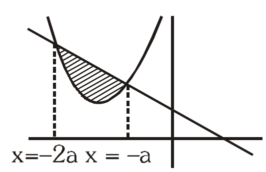
$= \frac{11}{8}$

12. $A = \int_0^{1/3}(x-1)^2dx+\int_{1/3}^{2/3}2x(1-x)dx+\int_{2/3}^{1}x^2dx$



15. $A = \int_{-a}^{-2a}\frac{a^2-ax-(x^2+2ax+3a^2)}{1+a^4}dx$

$= \frac{3}{2}\,\frac{a^3}{1+a^4}$



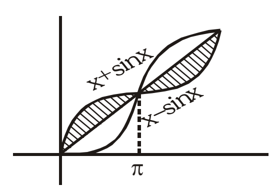
Now $f(a) = \frac{3}{2}\,\frac{a^3}{1+a^4}$

$\Rightarrow f'(a) = 0$

$\Rightarrow (1+a^4)\,3a^2-a^3\,4a^3 = 0$

$\Rightarrow \quad a_{min} = 0,\ a_{max} = 3^{1/4}$

17. $A = 4\int_0^{\pi}[x+\sin x-x)]dx$



2. According to question

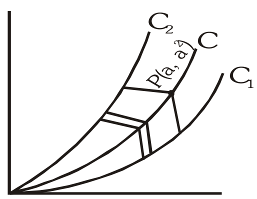
$$\int_0^{a^2}\left(-f^{-1}(y)+\sqrt{y}\right)\,dy = \int_0^{a}\left(x^2-\frac{x^2}{2}\right)dx$$

$\Rightarrow [f^{-1}(a^2) - a]\ 2a = -\ \dfrac{a^2}{2}$

$\Rightarrow f^{-1}(a^2) = \dfrac{3a}{4}$

$\Rightarrow f\left(\dfrac{3a}{4}\right) = a^2$

or $f(x) = \dfrac{16}{9}x^2$



3. $A_n = \displaystyle\int_0^{\pi/4}(\tan x)^n\,dx$

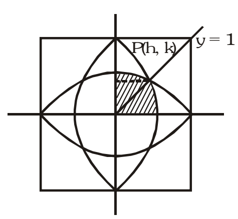
$A_n + A_{n-2} = \displaystyle\int_0^{\pi/4}\left[(\tan x)^n + (\tan x)^{n-2}\right]dx$

$= \displaystyle\int_0^{\pi/4}(\tan x)^{n-2}\sec^2 x\ \ dx = \left[\frac{t^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$

Also $A_{n+2} < A_n < A_{n-2}$

$\Rightarrow \dfrac{1}{n+1} < 2A_n < \dfrac{1}{n-1}$

4. Distance of point P from origin is less then distance of P from y = 1



$\sqrt{h^2+k^2} < k - 1\ ;\ \sqrt{h^2+k^2} < -\ k - 1$

$\Rightarrow x^2 + y^2 < (y - 1)^2\ ;\ x^2 + y^2 < y^2 + 2y + 1$

$\Rightarrow x^2 < -\ 2\left(y-\dfrac{1}{2}\right)\ ;\ x^2 < 2\ \left(y+\dfrac{1}{2}\right)$

similarly $y^2 < -\ 2\left(x-\dfrac{1}{2}\right)\ ;\ y^2 < 2\ \left(x+\dfrac{1}{2}\right)$

$\Rightarrow y = \dfrac{x^2-1}{-2}$ or $y = x = \dfrac{x^2-1}{-2}$

$\Rightarrow x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x = -1 \pm \sqrt{2}$

$$A = 8\int_0^{\sqrt{2}-1}\left[\frac{1-x^2}{2}-\sqrt{2}+1\right]dx + 4(\sqrt{2}-1)^2$$

$$= \frac{16\sqrt{2}-20}{3}$$

6. $f(x + 1) = f(x) + 2x + 1$

$\Rightarrow f''(x + 1) = f''(x) \qquad \forall\ x \in R$

Let $f''(x) = a \ \Rightarrow\ f'(x) = ax + b$

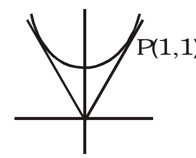
$\Rightarrow f(x) = \dfrac{ax^2}{2} + bx + c$

$\Rightarrow c = 1 \qquad [\because\ f(0) = 1]$

Now $f(x + 1) - f(x) = 2x + 1$

$$\Rightarrow \left[\frac{a}{2}(x+1)^2 + b(x+1) + c\right] - \left[\frac{ax^2}{2} + bx + c\right] = 2x+1$$

$\Rightarrow ax + \dfrac{a}{2} + b = 2x + 1$



on comparing we get a = 2,

or $\dfrac{a}{2} + b = 1 \Rightarrow b = 0$

$\therefore\ f(x) = x^2 + 1$ ... (i)

Now let equation of tangent be $y = mx$ ... (ii)

from (i) and (ii)

$x^2 - mx + 1 = 0 \Rightarrow m = \pm\ 2$

$\therefore$ tangent are $y = 2x$ or $y = -2x$

$$A = 2\int_0^1 (x^2+1-2x)dx = \frac{2}{3}$$

8. Curve $y = a - bx^2$ passes through the point (2, 1)

$\therefore\ a - 4b = 1$

$$A = 2\int_0^{\sqrt{a/b}}(a-bx^2)dx = 2\left[ax-\frac{bx^3}{3}\right]_0^{\sqrt{a/b}}$$

$$= \frac{4}{3}\frac{a^{3/2}}{\sqrt{b}} = \frac{4}{3}\ \frac{(1+4b)^{3/2}}{\sqrt{b}}$$

$A' = \dfrac{2}{3}\ \dfrac{\sqrt{1+4b}(8b-1)}{b^{3/2}} \ \Rightarrow\ A' = 0 \Rightarrow b = \dfrac{1}{8}$

$\Rightarrow A = 4\sqrt{3}$ sq. units

10. $$f(x) = \begin{cases} x^2+ax+b & ; \quad x<-1 \\ 2x & ; \quad -1\le x\le 1 \\ x^2+ax+b & ; \quad x>1 \end{cases}$$

$\because\ f(x)$ is continuous at $x = -1$ and $x = 1$

$\therefore\ (-1)^2 + a(-1) + b = -2$

and $2 = (1)^2 + a.\ 1 + b$

i.e., $a - b = 3$

and $a + b = 1$

on solving we get $a = 2,\ b = -1$

$$\therefore\ f(x) = \begin{cases} x^2+2x-1 & ; \quad x<-1 \\ 2x & ; \quad -1 \le x \le 1 \\ x^2+2x-1 & ; \quad x>1 \end{cases}$$

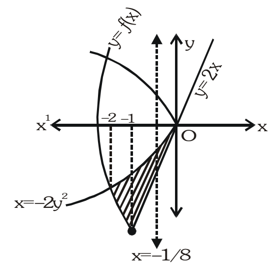
Given curves are

$y = f(x),\ x = -2y^2$ and $8x + 1 = 0$

solving $x = -2y^2,\ y = x^2 + 2x - 1\ (x < -1)$ we get $x = -2$.

Also $y = 2x,\ x = -2y^2$ meet at $(0, 0)$

and $\left(-\frac{1}{8}, -\frac{1}{4}\right)$



The required area is the shaded region in the figure.

$\therefore$ Required area

$$= \int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}} - (x^2+2x-1)\right]dx + \int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}} - 2x\right]dx$$

$$= \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - \frac{x^3}{3} - x^2 + x\right]_{-2}^{-1}$$

$$+ \left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3} - x^2\right]_{-1}^{-1/8} = \frac{257}{192} \text{ square units}$$

EXERCISE - 05 [A] JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ Put $y = vx \Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{v^2x^2-x^2}{2x.vx} = \frac{v^2-1}{2v}$

or $\frac{xdv}{dx} = \frac{v^2-1}{2v} - v$

$x\frac{dv}{dx} = \frac{-v^2-1}{2v}$ or $-\int\frac{2v\,dv}{1+v^2} = \int\frac{dx}{x}$

$-\log(1 + v^2) = \log x + c$

$\log x + \log(1 + v^2) = \log c$

$\log x.\left(1+\frac{y^2}{x^2}\right) = \log c$ or $x\left(\frac{x^2+y^2}{x^2}\right) = c$

$\frac{x^2+y^2}{x} = c$ or $x^2 + y^2 = cx$

2. $y = e^{cx}$

$\log y = cx$ ..... (i)

$\frac{1}{y}y' = c \Rightarrow y' = cy$

$c = \frac{y'}{y}$ put in equation (i) $\log y = \frac{y'}{y}.x$

or $y \log y = xy'$

3. Given $\frac{dy}{dx} = \frac{y-1}{x(x-1)}$ or $\int\frac{dy}{y-1} = \int\frac{dx}{x(x+1)}$

$\log(y - 1) = \log\left(\frac{x}{x+1}\right) + \log C$

or $y - 1 = \frac{cx}{x+1}$ ..... (i)

Equation (i) passes through (1, 0)

$-1 = \frac{C}{2} \Rightarrow C = -2$ Put in (i)

$(y - 1) = \frac{-2x}{x+1}$ $(y - 1)(x + 1) + 2x = 0$

4. Equation of given parabola is $y^2 = Ax + B$ where A and B are parameters

$2y\frac{dy}{dx} = A$ $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

This is the equation of given parabola order = 2, degree 1

5. $(1 + y^2) = (e^{\tan^{-1}y} - x)\frac{dy}{dx}$ or $(1+ y^2)\frac{dx}{dy} + x + e^{\tan^{-1}y}$

$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{(1+y^2)}$

Now I.F. $= e^{\int\frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

$\therefore$ solution $x(e^{\tan^{-1}y}) = \int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy + C$

$xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2} + C$

or $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + K$

6. Given family of curves is

$x^2 + y^2 - 2ay = 0$ ..... (1)

$2x + 2yy' - 2ay' = 0$ ..... (2)

Now put the value of 2a from (1) to in (2)

$$2x + 2yy' - \frac{x^2+y^2}{y}.y' = 0$$

$2xy + (y^2 - x^2)y' = 0$ or $(x^2 - y^2)y' = 2xy$

7. $ydx + (x + x^2y)dy = 0$ $ydx + xdy = -x^2y\,dy$

$$\int \frac{d(xy)}{(xy)^2} = -\int \frac{dy}{y} \Rightarrow -\frac{1}{xy} = -\log y + c$$

$$\Rightarrow \frac{-1}{xy} + \log y = c$$

8. $y^2 = 2c(x + \sqrt{c})$ ..... (1)

$y^2 = 2cx + 2c\sqrt{c}$

$2y\frac{dy}{dx} = 2c \Rightarrow yy_1 = C$ Put in equation(1)

$\Rightarrow y^2 = 2yy_1(x + \sqrt{yy_1})$

$y^2 = -2yy_1x = 2yy_1\sqrt{yy_1}$ or $(y^2 - 2yy_1x)^2 = 4y^3y_1^3$

Degree = 3 order = 1

9. $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x}+1\right)$ which is homogeneous equ.

Put $y = vx$, $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left(\log\frac{vx}{x}+1\right)$$

$$\frac{xdv}{dx} = v(\log v + 1) - v = v\log v + v - v$$

$$\int\frac{dv}{v\log v} = \int\frac{dx}{x} \Rightarrow \log(\log v) = \log x + \log c$$

$$\Rightarrow \log\frac{y}{x} = cx$$

10. Given $Ax^2 + By^2 = 1$ Divide by B

$\frac{A}{B}x^2 + y^2 = \frac{1}{B}$ Differentiate w.r.t x

$2x\frac{A}{B} + 2y\frac{dy}{dx} = 0$ ..... (i)

Again Differentiate w.r.t. x

$2\frac{A}{B} + 2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] = 0$ ..... (ii)

Put $\frac{A}{B} = -\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]$ in equation (i)

$$-2x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] + 2y\frac{dy}{dx} = 0$$

or $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

It have second order and first degree.

11. Let the centre of circle is (h, 0) and radius will be also h

$\therefore$ equation of circle $(x - h)^2 + (y - 0)^2 = h^2$

$\Rightarrow x^2 - 2hx + h^2 + y^2 = h^2$

$\Rightarrow x^2 - 2hx + y^2 = 0$ ..... (i)

Equation (i) passes through origin differentiating it w.r.t. x

$2x-2h+2y\frac{dy}{dx} = 0 \Rightarrow h = x + y\frac{dy}{dx}$ put in equation (i)

$$x^2 - 2x\left(x+y\frac{dy}{dx}\right) + y^2 = 0$$

$$\Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

12. $\frac{dy}{dx} = 1 + \frac{y}{x}$ put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = 1 + \frac{vx}{x} \Rightarrow x\frac{dv}{dx} = 1$

$\int dv = \int\frac{dx}{x} \Rightarrow v = \log x + c$ or $\frac{y}{x} = \log x + c$ ... (i)

Given $y(1) = 1 \Rightarrow 1 = \log 1 + c \Rightarrow c = 1$ put (i)

$y = x\ell nx + x$

13. Equation of circle $(x - h)^2 + (y - 2)^2 = 25$ .....(i)

Differentiate w.r.t. x

$$2(x - h) + 2(y - 2)\frac{dy}{dx} = 0$$

$(x - h) = -(y - 2)\frac{dy}{dx}$ put in (i)

$$(y - 2)^2\left(\frac{dy}{dx}\right)^2 + (y - 2)^2 = 25$$

or $(y - 2)^2(y')^2 + (y - 2)^2 = 25$

14. Given $y = f(x)$

Tangent at point P(x, y)

$$Y - y = \left(\frac{dy}{dx}\right)_{(x,y)}(X - x)$$

Now y-intercept $\Rightarrow Y = y - x\frac{dy}{dx}$

Given that, $y - x\frac{dy}{dx} = x^3$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -x^2$ is a linear differential equation

with I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ell nx} = e^{\ell n\left(\frac{1}{x}\right)} = \frac{1}{x}$

Hence, solution is $\frac{y}{x} = \int -x^2 . \frac{1}{x} dx + C$

or $\frac{y}{x} = -\frac{x^2}{2} + C$

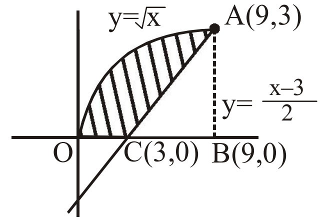
Given $f(1) = 1$

Substituing we get, $C = \frac{3}{2}$

so $y = -\frac{x^3}{2} + \frac{3}{2}x$

Now $f(-3) = \frac{27}{2} - \frac{9}{2} = 9$

15.



intersection point $\sqrt{x} = \frac{x-3}{2}$

$\Rightarrow \quad x - 2\sqrt{x} - 3 = 0$

$\sqrt{x} = 3, -1 \quad \Rightarrow x = 9$

Required Area $= \int_0^9 x^{1/2} \, dx$ – area of ΔABC

$= \left|\frac{2}{3}x^{3/2}\right|_0^9 - \frac{1}{2}.6.3 = 18 - 9 = 9$

3. The given curves are $y = x^2$
which is an upward parabola with vertex at (0, 0)
$y = |2 - x^2|$

or $y = \begin{cases} 2 - x^2 & \text{if} \quad -\sqrt{2} < x < \sqrt{2} \\ x^2 - 2 & \text{if} \quad x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$

or $x^2 = -(y - 2)$; $\quad -\sqrt{2} < x < \sqrt{2}$ ... (2)

a downward parabola with vertex at (0, 2)

$x^2 = y + 2$; $\quad x < -\sqrt{2}$, $x > \sqrt{2}$ ... (3)

On upward parabola with vertex at (0, –2)

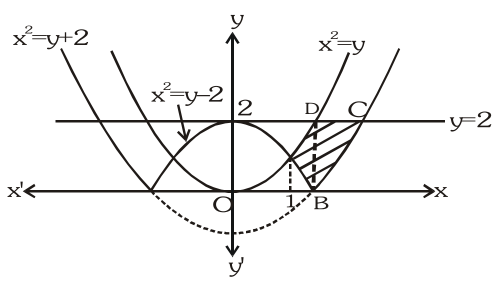
$y = 2$ ... (4)

Straight line parallel to x-axis

$x = 1$ ... (5)

Straight line parallel to y-axis

The graph of these curves is as follows.



$\therefore$ Required area = BCDEB

$= \int_1^{\sqrt{2}} [x^2 - (2 - x^2)]dx + \int_{\sqrt{2}}^{2} [2 - (x^2 - 2)]dx$

$= \int_1^{\sqrt{2}} (2x^2 - 2)dx + \int_{\sqrt{2}}^{2} (4 - x^2)dx = \left(\frac{20}{3} - 4\sqrt{2}\right)$ sq. units

8. We have, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$

$\Rightarrow 4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$
$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3b$
$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

Consider the equation

$4x^2 f(-1) + 4xf(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]\, x + f(2) = 0$

Then clearly this equation is satisfied by

$x = a, b, c$

A quadratic equation satisfied by more than two values of x means it is an identity and hence

$4f(-1) - 3 = 0 \quad \Rightarrow \quad f(-1) = 3/4$
$4f(1) - 3 = 0 \quad \Rightarrow \quad f(1) = 3/4$
$f(2) = 0 \quad \Rightarrow \quad f(2) = 0$

Let $f(x) = px^2 + qx + r$ [$f(x)$ being a quad. equation]

$f(-1) = \frac{3}{4} \quad \Rightarrow \quad p - q + r = \frac{3}{4}$

$f(1) = \frac{3}{4} \quad \Rightarrow \quad p + q + r = \frac{3}{4}$

$f(2) = 0 \quad \Rightarrow \quad 4p + 2q + r = 0$

Solving the above we get $q = 0$, $p = \frac{-1}{4}$, $r = 1$

$\therefore f(x) = -\frac{1}{4}x^2 + 1$

It's maximum value occur at $f'(x) = 0$

i.e., $x = 0$ then $f(x) = 1$ $\quad \therefore$ V(0, 1)

A (–2, 0) is the pt. where curve meet x-axis

Let B be the pt. $\left(h, \frac{4 - h^2}{4}\right)$

As $\quad \angle AVB = 90$

$m_{AV} \quad m_{BV} = -1$

$\Rightarrow \quad \frac{1}{2} \times \left(\frac{-h}{4}\right) = -1$

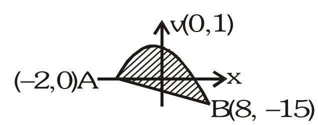
$\Rightarrow \quad h = 8 \quad \therefore$ B(8, – 15)

Equation of chord AB is

$y + 15 = \frac{0 - (-15)}{-2 - 8}(x - 8)$

$\Rightarrow 3x + 2y + 6 = 0$

Required area is the area of shadded region given by



$= \int_{-2}^{8} \left[\left(-\frac{x^2}{4} + 1\right) - \left(\frac{-6 - 3x}{2}\right)\right]dx$

$= \frac{125}{3}$ sq. units.

9. (c) By inspection, the point of intersection of two curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is (1, 0)

For first curve $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1} \log 3 \log x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_1$

For second curve $\frac{dy}{dx} = x^x (1 + \log x)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = 1 = m_2$

$\because \quad m_1 = m_2 \Rightarrow$ two curves touch each other

$\Rightarrow \quad$ angle between them is 0

$\therefore \quad \cos\theta = 1$

10. $y^3 - 3y + x = 0$

$3y^2y' - 3y' + 1 = 0$ $\qquad y' = \dfrac{-1}{3(y^2-1)}$

$f(-10\sqrt{2}) = 2\sqrt{2}$

$f'(-10\sqrt{2}) = -\dfrac{1}{3(7)} = -\dfrac{1}{21}$

$6y(y')^2 + 3y^2y'' - 3y'' = 0$

$y'' = -\dfrac{2y(y')^2}{y^2-1}$

$f''(-10\sqrt{2}) = \dfrac{-2(2\sqrt{2})}{441\times 7} = \dfrac{-4\sqrt{2}}{7^3 3^2}$

11. $\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$

$= bf(b) - af(a) + \int_a^b \dfrac{x}{3\left[(f(x))^2-1\right]}dx$

$= \int_a^b \dfrac{x}{3\left[(f(x))^2-1\right]}dx + bf(b) - af(a)$

12. $\int_{-1}^{1} g'(x)dx = g(1) - g(-1)$

Now $g(1) = -(g(-1))$

(as $g'(x)$ is an even function)

so $\int_{-1}^{1} g'(x)dx = 2g(1) = -2g(-1)$

13. Area $= \int_0^{\pi/4}\left(\sqrt{\dfrac{1+\sin x}{\cos x}} - \sqrt{\dfrac{1-\sin x}{\cos x}}\right)dx$

$= \int_0^{\pi/4} \dfrac{\left(\cos\dfrac{x}{2}+\sin\dfrac{x}{2}\right)-\left(\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)}{\sqrt{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}}dx$

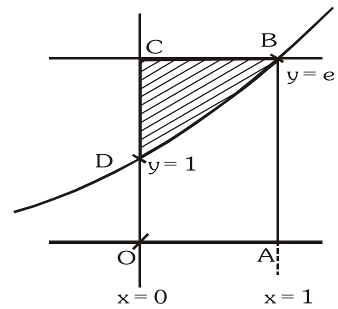
$= \int_0^{\pi/4} \dfrac{2\sin\dfrac{x}{2}}{\sqrt{\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}}}dx = \int_0^{\pi/4} \dfrac{2\tan\dfrac{x}{2}}{\sqrt{1-\tan^2\dfrac{x}{2}}}dx$

Let $\tan\dfrac{x}{2} = t$

$\sec^2\dfrac{x}{2}\,dx = 2dt \Rightarrow dx = \dfrac{2dt}{(1+t^2)}$

$\therefore$ Area $= \int_0^{\sqrt{2}-1} \dfrac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

14. $A = \int_1^e \ln y\, dy$



Apply

$= \int_1^e \ln(e+1-y)dy$

A = ar (OABC) − ar (OABD)

$= e - \int_1^e e^x dx$.

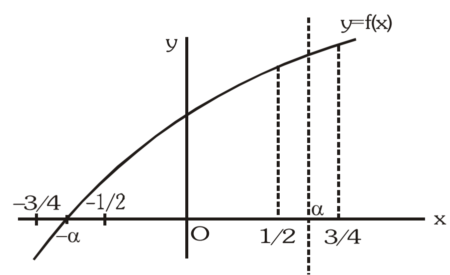
15. $\because f'(x) = 2 + 6x + 12x^2 > 0 \ \forall\ x \in R$

$\therefore$ f(x) is strictly increasing in R

$\because f(0) = 1,\ f(-1) = -2,\ f\left(-\dfrac{1}{2}\right) = \dfrac{1}{4}\ \ \&\ f\left(-\dfrac{3}{4}\right) = -\dfrac{1}{2}$

$\therefore$ f(x) = 0 has only one real root lying in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$

16.



Let real root is $-\alpha$

$\Rightarrow\ t = |s| = \alpha$

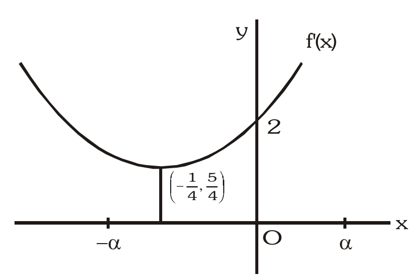
Required area

$A = \int_0^{\alpha} f(x)dx \quad \& \quad \int_0^{1/2} f(x)dx < A < \int_0^{3/4} f(x)dx$

$\Rightarrow \left|x + x^2 + x^3 + x^4\right|_0^{1/2} < A < \left|x + x^2 + x^3 + x^4\right|_0^{3/4} < \left|4x\right|_0^{3/4}$

$\Rightarrow \dfrac{15}{16} < A < 3$

17.



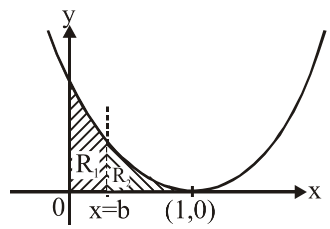
$f'(x) = 2\,(6x^2 + 3x + 1)$

$\Rightarrow$ f'(x) is decreasing in $\left(-\alpha, -\dfrac{1}{4}\right)$ increasing

in $\left(-\dfrac{1}{4}, \alpha\right)$

or f'(x) is decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

18. (a) $\because$ $R_1 - R_2 = \frac{1}{4}$



$\Rightarrow \int_0^b (1-x)^2\,dx - \int_b^1 (1-x)^2\,dx = \frac{1}{4}$

$\Rightarrow -\left(\frac{(1-x)^3}{3}\right)_0^b + \left(\frac{(1-x)^3}{3}\right)_b^1 = \frac{1}{4}$

$\Rightarrow -\left\{\frac{(1-b)^3}{3} - \frac{1}{3}\right\} - \frac{(1-b)^3}{3} = \frac{1}{4}$

$\Rightarrow \frac{1}{3} - \frac{2}{3}(1-b)^3 = \frac{1}{4}$ $\Rightarrow \frac{2}{3}(1-b)^3 = \frac{1}{12}$

$\Rightarrow (1-b)^3 = \frac{1}{8}$ $\Rightarrow 1 - b = \frac{1}{2}$

$\Rightarrow b = \frac{1}{2}$

(b) $R_2 = \int_{-1}^{2} f(x)dx$, $R_1 = \int_{-1}^{2} x f(x)dx$

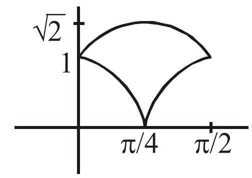
$= \int_{-1}^{2} (1-x)f(1-x)dx$ $\left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$

$= \int_{-1}^{2} (1-x)f(x)dx$ (given $f(x) = f(1-x)$)

$= \int_{-1}^{2} f(x)dx - \int_{-1}^{2} x f(x)dx$

or $R_1 = R_2 - R_1$ $\Rightarrow$ $2R_1 = R_2$

19. $y = \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$



$y = |\cos x - \sin x| = \sqrt{2}\left(\cos\left(x + \frac{\pi}{4}\right)\right)$

Area

$= \int_0^{\pi/4} \left[(\sin x + \cos x) - (\cos x - \sin x)\right]dx$

$+ \int_{\pi/4}^{\pi/2} \left[(\sin x + \cos x) - (\sin x - \cos x)\right]dx$

$= \int_0^{\pi/4} 2\sin x\,dx + \int_{\pi/4}^{\pi/2} 2\cos x\,dx$

$= \left[-2\cos x\right]_0^{\pi/4} + \left[2\sin x\right]_{\pi/4}^{\pi/2}$

$= 2\sqrt{2}\left(\sqrt{2} - 1\right)$